Lectures on Really Useful Ad Hoc Macroeconomics

Willem H. Buiter[1]
Faculty of Economics and Politics, University of Cambridge
and Monetary Policy Committee, Bank of England[2]

29 November 1999

These lecture notes are provisional and under constant revision. I welcome notifications of type I and type II errors.

[2] The views and opinions expressed are those of the author only. They do not necessarily represent the views and opinions of the Bank of England or of the other members of the Monetary Policy Committee

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Chapter 1

Introduction

These lecture notes are based on lectures given at the London School of Economics, Yale University and the University of Cambridge since the early 1980's. They are aimed at advanced undergraduates and Masters level students in economics. The main guiding principle for including and excluding topics has been their relevance for the evaluation and design of macroeconomic policy. Before June 1997, the policy relevance of the material was informed mainly by my advisory work for the International Monetary Fund, the World Bank, the Inter-American Development Bank, the European Bank for Reconstruction and Development, the European Community (and later the European Union), and the national governments, parliaments and central banks in a number of industrial countries (including the UK and the Netherlands), developing countries and transition economies. Since June 1997, my work as an independent member of the Monetary Policy Committee of the Bank of England has shaped and altered my views on what constitutes useful macroeconomics.

The principle of policy relevance has been overridden at times when a particular bit of theory that, in my view, had no practical policy relevance, was so prevalent in the professional literature and in current academic discourse, that students needed to be aware of them in order to participate effectively in the policy debate. Among these irrelevant but omnipresent items are the analysis of economies without nominal wage and price rigidities, the analysis of closed economic systems (still very common, especially in the USA), the representative agent model of consumer demand and portfolio selection, and real business cycle theory.

Every lecturer begs, steals and borrows ideas, models and theories from fellow economists past and present. I have tried to acknowledge the main sources of most of the theories and models expounded in these notes, but it is effectively impossible to give full and comprehensive credit everywhere

it might be due. I make no claims of originality for the ideas and analyses contained in these notes. Where new material is presented (as in the analysis of the sacrifice ratio in Chapter 2, and in the analysis of the role of forward-looking financial markets in simple macromodels in Chapter 3), it constitutes a straightforward extension of the existing literature.

My time as an MPC member has had a significant impact on my views on what constitutes relevant macroeconomics. One area in which this is reflected is in the treatment of monetary policy. While most monetary analysis is based on the assumption that some monetary aggregate is the instrument controlled by the central bank, the reality is that central banks set a short nominal interest rate, with the money stock determined endogenously. Throughout these notes, I therefore focus on nominal interest rules for monetary policy although, again in order to preserve the links with the existing literature, money stock rules or monetary growth rules are also considered.

The fact that national economies are open to trade in goods, services and financial claims means (and may also be open to international migration of workers, consumers and firms) means that for national policy makers an open economy perspective is essential. The attention paid to open economy issues in these lectures therefore greatly exceeds that characteristic of most macroeconomics treatises, especially those written in the USA.

Even during the heyday of the New Classical Macroeconomics (the late seventies and early eighties), central banks were a bastion of Keynesianism, in the sense that every central banker believed that nominal rigidities were a fact of life (except under conditions of hyper-inflation) and that monetary policy could therefore have significant, if temporary, effects on real economic activity. A key challenge for theorists, modelers and policy analysts has been to find a characterisation of these nominal rigidities that is invariant under the kind of changes in policy rules, in the external environment and in other exogenous processes that are likely to occur. This search for a representation of nominal rigidities, that is, persistence in wage and price inflation, that is not ambushed fatally by the Lucas critique is complicated by the fact that there are no satisfactory 'deep structural' theories of nominal rigidities. New Keynesian 'menu cost' theories simply assume the answer: there are real costs of changing nominal prices. One would really like to see some discrete gap between the assumption and the conclusion.

The fundamental reason why there is no satisfactory theory of nominal rigidities is that there is no satisfactory theory of money. It has long been recognised that there is no convincing theory of the emergence and use of a general means of payment and medium of exchange. Hence the recourse to such theoretical abominations as (fiat) money in the direct utility function, money as a constraint on exchange (the cash-in-advance models), money in

the production function and money in a 'shopping function', economising on time and other real resources. Allais-Baumol-Tobin models of money demand and other inventory-theoretic models of money demand (such as the sS model of Miller and Orr) purport to explain the holding of a rate of return dominated transactions medium by postulating (plausible) transactions costs associated with switching between the transactions medium and the superior stores of value. Fundamentally, however, they are very strict cash-in-advance models, since they assume that money is required for consumer purchases (Allais-Baumol-Tobin) or business purchases and sales (Miller-Orr).

Even more damaging than the absence of a viable theory of the means of payment and medium of exchange is the absence of a theory of the numéraire. While it seems entirely plausible and sensible that the numéraire. should be the same as the medium of exchange (despite the odd exceptions like the Guinea), conventional economics cannot explain why money rather than oranges are the numéraire.. If oranges were the numéraire. and if there were menu costs, the economy would be characterised by rigidities of prices and wages in terms of oranges. With such orange rigidities, monetary policy would be emasculated. Explaining why money is the numéraire. requires an appeal to some notion of bounded rationality. It is hard to compute, calculate and re-calculate. This bounded rationality perspective also suggest another reason why inflation or deflation is costly: there are costs associated with the mental exercise of measuring with a yardstick whose length if forever changing. Such mental menu costs cannot be accommodated in conventional economic analysis.

When faced with the absence of a theory of the medium of exchange, of a theory of the numéraire. and of a theory of nominal rigidities, the economists can do three things. The first is to leave the profession in despair. The second is to argue as follows: I cannot explain nominal rigidities in my favoured paradigm. Therefore I will proceed as if they do not exist. This is the conventional fresh-water approach, coming out of Chicago, Minnesota, Rochester, Carnegie-Mellon and their intellectual subsidiaries. I have always been astonished by this attitude, which amount to saying: if I don't understand something, it cannot exist.

The third approach is more intellectually modest and therefore naturally appeals to me. It argues as follows. I do not have a satisfactory explanation of nominal rigidities.. However, they appear to be there, and they appear to matter for macroeconomic policy. I will therefore try and capture the apparent empirical regularities as best I can, using some ad-hoc adjustment mechanism. I will use whatever structure economic theory does provide to constrain the set of permissible adjustment mechanism. Absence of money illusion is one such a-priori restriction. I then proceed to keep my fingers

crossed and hope that the nominal price and wage dynamics I come up with are invariant under the kinds of interventions performed by policy, the external environment, nature or any other factor. That is, I hope that the Lucas critique won't strike, but stand ready to revise my view of money wage and price dynamics whenever the assumed process appear to break down in a systematic manner. The results of this ad-hoc, eclectic approach can be found in Chapter 2 on short-run aggregate supply behaviour and wage-price dynamics.

The next point, which generalises the previous one, concerns the relative merits of *ad-hoc* decision rules versus behaviour derived from explicit optimising choices by economic agents. Here again, I have found that an attitude of enlightened agnosticism is optimal. Optimising foundations are attractive because one has to be fully explicit about the objectives of the optimising agent and about the environment within which choice is exercised (technology, budget constraints, market structure and other aspects of the competitive environment, information etc.). At the same time, it should be recognised that this is by no means a sufficient condition for obtaining sensible behavioural rules. The optimisation of an arbitrary objective function subject to implausible constraints is unlikely to result in useful economic models. I would also argue that optimising foundations are also not necessary for the derivation of useful decision rules and the construction of interesting equilibrium configurations. One fundamental reason for this is that economic agents do not know the environment (the model) within which they operate. This is a major conceptual and, I would argue, practical weakness of rational expectations models, which assume model-consistent expectations.. Agents have to learn about their environment. The concept of 'rational learning' is an oxymoron.

Some optimising models, such as the overlapping generations model of consumer behaviour, in both its Allais-Samuelson version and in its Yaari-Blanchard version, are extremely useful for getting a handle on intergenerational distributional issues, including the intergenerational redistribution associated with the choice of government borrowing vs. current taxes. Extensive use is made of both kinds of OLG models in the chapter dealing with private consumption behaviour. Unlike the representative agent model, the OLG model has the minimal amount of consumer heterogeneity (old vs. young, currently alive vs. unborn) required for a first stab at dynamic fiscal policy issues. Even the OLG model, however, overestimates the ability of most households to borrow against expected future labour income. A simple, *ad-hoc*, modification that allows for liquidity-constrained consumers is to postulate that a fraction α of the population follows the life-cycle model while a fraction $(1-\alpha)$ simply consumes its current income. One can ra-

tionalise such 'myopic' behaviour by assuming that these consumers have phenomenally high pure rates of time preference, but that would be no more than vacuous relabeling. Models of optimising consumer choice and portfolio selection also provide some useful 'first insight' on the road to an empirically implementable representation of consumption and asset demand. This is true despite the heroic empirical failure of every version of the capital asset pricing model. The effect of uncertainty on investment in the presence of costly reversibility and expandability of investment projects likewise can be analysed conveniently in an optimising framework.

It has become fashionable in the last decade or so among fresh water economists to be snooty about the IS-LM, aggregate demand-aggregate supply model. This attitude seems quite misplaced. The IS-LM, AD-AS model is a very useful *portmanteau* on which it is possible to hang a great variety of useful small macroeconomic models, ad-hoc, optimising, evolutionary or what not. This extremely flexible and convenient workhorse figures prominently in what follows.

The only sensible way to proceed is to be catholic in one's tastes and to try anything that appears to have a reasonable chance of working, in the sense of producing stable relationships between observables. Decision rules can be derived from optimising behaviour, satisficing behaviour, prima-facie plausible rules of thumb, from behavioural psychology, from socio-biology, from the swarming behaviour of the africanised bee, from introspection or by looking out of the window. Equilibrium concepts constraining the relationships between individual agents or groups of agents can likewise varied and non-standard.

Being an accountant at heart, I have a strong preference, when modeling market economies, to embed agents' market transactions in a comprehensive set of budget constraints and balance sheet constraints. As a practical matter, however, it turns out that there are large numbers of simple and small dynamic macroeconomic models that yield useful insights without the benefit of comprehensive stock-flow accounting.

The issue of equilibrium vs. disequilibrium models is a purely semantic, that is, vacuous one. Equilibrium, like rationality, is such a weak concept that it does not impose any serious constraints on observable behaviour. An equilibrium is any configuration of endogenous variables that satisfies a set of conditions (often expressed as equalities or inequalities) set by the modeler.. It can be a competitive equilibrium, even an Arrow-Debreu equilibrium; it can be a Nash equilibrium or a Nash bargaining equilibrium. It can be an equilibrium with or without rationing, and with or without rational (model-consistent) expectations. Thus an ad-hoc Keynesian models, such as the IS-LM fixed price model, is an equilibrium model, simply because it solves

consistently for output and interest rate combinations that satisfy the monetary equilibrium condition and the condition that expenditure equals output, when output is demand-constrained and the price level is fixed.

Useful macroeconomic models manage to come up with stable relationships between observable aggregates that reflect the importance of the underlying microeconomic heterogeneity of households (as consumers, financial actors and workers), firms and financial intermediaries, without exposing the user to the non-transparent mess that is the almost unavoidable by-product of the explicit aggregation of individual behaviour. Physicists can study the behaviour of gaseous clouds without having to build up that behaviour from the explicit aggregation of the behaviour of the elementary particles that make up the gas molecules and clouds. Economists can therefore surely be forgiven for once again going boldly where few seem to want to go any longer, by postulating and testing relationships between aggregates without worrying unduly about the connection between the aggregate relationships and the behaviour of the individual Joes and Maries. That, at any rate, is what I do routinely and unapologetically, in these lectures. Optimising foundations should be a means to the end of more robust, models that can be used for policy analysis, not an end in themselves.

Chapter 2

The Static Closed Economy AD-AS Model

2.1 The Fixed Price Case

The IS-LM, aggregate demand - aggregate supply model is a short run, temporary, momentary or instantaneous equilibrium model. It treats as given all expectations of future endogenous variables, the predetermined financial and physical asset stocks and the real price of equity.

In the closed economy version of the model there are three kinds of agents: *households*, *firms* and the *government*. Households are competitive in all markets in which they operate. Firms are competitive in the labour market and in the financial markets. In the Keynesian or fixed-price version of the model they are rationed in the markets for the goods they sell. In the flexible price level version, firms are competitive in the goods market also. Households consume, supply labour, make portfolio choices, pay taxes and enjoy the benefits of public spending. Households own all factors of production, including the enterprise sector of the economy. In the Keynesian or fixed-price version of the model, output and employment are demand-determined, and the labour supply decision can be ignored. Firms or enterprises hire labour, rent capital services (or use the services of capital they themselves own), produce and sell output, engage in investment (fixed and inventory) and finance this investment through some combination of retained earnings, new equity issues or corporate borrowing. For simplicity, it is assumed that firms do not pay taxes. The IS-LM, AD-AS model is most easily interpreted as one in which all investment is financed by new equity issues. We assume this in what follows. The government, which is the consolidated general government and central bank, spends on public consumption and investment, raises tax

revenues and other current revenues, pays transfer payments and subsidies, pays interest on its public debt and finances its financial deficit by printing base money or by borrowing. Its behaviour is exogenous.

The model has one homogeneous, malleable commodity, Y, which can be used interchangeably for private consumption, C, public consumption, C^g, private investment, I, and public investment, I^g. The money price of the commodity (the general price level in this one-good model) is P. The short (one-period or instantaneous) interest rate in terms of the good (the short real interest rate) is r.

There are three financial stores of value: non-interest-bearing base money, M, a liability of the government; interest-bearing public debt, B, assumed to be nominally denominated and with a one-period maturity; and real capital, K. The same symbol, K, is used to denote both the physical capital stock that is an argument in the production function, and the ownership claim to this physical capital stock, or equity. The one-period or instantaneous nominal rate of interest on the government bond is i, and the price of a unit of installed capital (and of a share of equity) in terms of output (the real price of capital or Tobin's q) is q. We also define the following further notation:

H : demand for real money balances
T : real taxes net of transfers
$\overline{T}$: income-independent taxes net of transfers
τ : marginal GDP tax rate
i_m :nominal interest rate on money
$r_m \equiv i_m - \hat{\pi}$: real rate of return on money
r_E :real rate of return on equity (ownership claims on real capital).
$\hat{\pi}$: expected proportional rate of inflation
π : proportional rate of inflation; $\pi \equiv \frac{\Delta P}{P}$
Y_d : real private disposable income
A : Nominal private financial wealth
$a \equiv \frac{A}{P}$: real private financial wealth
W : money wage
$w \equiv \frac{W}{P}$: real wage
L : employment

:

We start with the two following equilibrium conditions for, respectively, the goods market 2.1 and the demand for and supply of money 2.2.

$$Y = C + I + G \tag{2.1}$$

$$\frac{M}{P} = H \tag{2.2}$$

We define the following behavioural equations:

$$\begin{aligned} C &= c(Y_d, r, a) \\ 0 &< c_{Y_d} < 1;\ c_r \leq 0;\ c_a \geq 0 \end{aligned} \tag{2.3}$$

The marginal propensity to consume, c_{Y_d} is positive but less than one; the effect of the real interest rate on consumption, c_r is non-positive and the wealth effect on consumption, c_a is non-negative.

$$Y_d \equiv Y - T \tag{2.4}$$

$$\begin{aligned} T &= \bar{T} + \tau Y \\ 0 &< \tau < 1 \end{aligned} \tag{2.5}$$

Taxes net of transfers rise less than one-for-one with income.

$$r \equiv i - \hat{\pi} \tag{2.6}$$

$$a \equiv \frac{M + B}{P} + qK \tag{2.7}$$

$$\begin{aligned} I &= \iota(r, Y, K) \\ \iota_r &\leq 0;\ \iota_Y \geq 0;\ \iota_K \leq 0 \end{aligned} \tag{2.8}$$

A higher real interest rate does not raise investment, $\iota_r \leq 0$. Higher GDP does not lower investment, $\iota_Y \geq 0$, and the higher the capital stock, the lower the rate of investment, $\iota_K \leq 0$.

$$\begin{aligned} H &= h(i, Y, a) \\ h_i &\leq 0;\ h_Y > 0;\ 0 \leq h_a \leq 1 \end{aligned} \tag{2.9}$$

A higher opportunity cost of holding money does not increase the demand for money, $h_i \leq 0$. A higher income level (a proxy for the transactions performed by money, raises the demand for money, $h_Y > 0$. For portfolio diversification reasons,a higher stock of financial wealth increases the demand for money, but less than one-for-one, $0 \leq h_a \leq 1$.

Why is the nominal interest rate the real opportunity cost of holding money and why is there only one rate of return argument, i, in the money demand function, when there are three financial assets, money, government bonds and claims to real capital, in the model?

The underlying money demand function can be written as follows

$$H = h(r_m, r, r_E, Y, a)$$

If the three stores of value are gross substitutes, we have $h_{r_m} > 0;\ h_r < 0$ and $h_{r_E} < 0$.

The first simplifying assumption is that government bonds and equity are perfect substitutes as stores of value, that is,

$$r = r_E$$

The money demand function therefore becomes

$$H = h(r_m, r, Y, a)$$

with $h_{r_m} > 0$ and $h_r < 0$.

The second simplifying assumption is that the demand for money depends only on the difference between the real rates of return on bonds and on money, that is,

$$H = h(r - r_m, Y, a)$$

with $h_1 < 0$.

By definition,

$$\begin{aligned} r &\equiv i - \hat{\pi} \\ r_m &\equiv i_m - \hat{\pi} \end{aligned}$$

The money demand function therefore becomes

$$H = h(i - i_m, Y, a)$$

Narrow money is non-interest-bearing, that is, $i_m = 0$. This gives us the money demand function we are working with. Note that wider monetary aggregates, ($M1$, $M2$, $M3$, $M4$ etc.) are to a large extent interest-bearing, and that the nominal own rate of return on broad money is to a large extent market-determined, and tends to move in the same direction as the nominal yield on bonds.

Substituting the behavioural equations and identities into the two equilibrium conditions we get:

$$c\left(Y(1-\tau) - \bar{T}, i - \hat{\pi}, \frac{M+B}{P} + qK\right) + \iota(i - \hat{\pi}, Y, K) + G = Y \quad (2.10)$$

$$h(i, Y, \frac{M+B}{P} + qK) = \frac{M}{P} \quad (2.11)$$

We classify the variables as follows:

Exogenous or predetermined: P, M, B, K, q, G , τ, $\bar{T}$, $\hat{\pi}$

Endogenous: Y and i

Equations 2.10 and 2.11 define two implicit non-linear equations determining the equilibrium values of Y and i.

$$f^{IS}(Y, i; P, M, B, K, q, G, \tau, \bar{T}, \hat{\pi}) = 0 \quad (2.12)$$

$$f^{LM}(Y, i; P, M, B, K, q, G, \tau, \bar{T}, \hat{\pi}) = 0 \quad (2.13)$$

Assuming that the conditions for the implicit function theorem to hold are satisfied by 2.12 and 2.13, we can obtain from the non-linear structural equations 2.12 and 2.13 a reduced form linear approximation as follows:

(1) Totally differentiate 2.12 and 2.13 at some equilibrium $Y^*, i^*; P^*, M^*, B^*, K^*, q^*,$

$G^*, \tau^*, \bar{T}^*, \hat{\pi}^*$.
This gives

$$f_Y^{IS}dY + f_i^{IS}di + f_P^{IS}dP + f_M^{IS}dM + f_B^{IS}dB + f_K^{IS}dK + f_q^{IS}dq + f_G^{IS}dG \\ + f_\tau^{IS}d\tau + f_{\bar{T}}^{IS}d\bar{T} + f_{\hat{\pi}}^{IS}d\hat{\pi} = 0$$

$$f_Y^{LM}dY + f_i^{LM}di + f_P^{LM}dP + f_M^{LM}dM + f_B^{LM}dB + f_K^{LM}dK + f_q^{LM}dq + f_G^{LM}dG \\ + f_\tau^{LM}d\tau + f_{\bar{T}}^{LM}d\bar{T} + f_{\hat{\pi}}^{LM}d\hat{\pi} = 0$$

Rearrange this as follows (endogenous variables on the LHS, exogenous and predetermined variables on the RHS):

$$\begin{bmatrix} f_Y^{IS} & f_i^{IS} \\ f_Y^{LM} & f_i^{LM} \end{bmatrix} \begin{bmatrix} dY \\ di \end{bmatrix} =$$

$$-\begin{bmatrix} f_P^{IS} & f_M^{IS} & f_B^{IS} & f_K^{IS} & f_q^{IS} & f_G^{IS} & f_\tau^{IS} & f_{\bar{T}}^{IS} & f_{\hat{\pi}}^{IS} \\ f_P^{LM} & f_M^{LM} & f_B^{LM} & f_K^{LM} & f_q^{LM} & f_G^{LM} & f_\tau^{LM} & f_{\bar{T}}^{LM} & f_{\hat{\pi}}^{LM} \end{bmatrix} \begin{bmatrix} dP \\ dM \\ dB \\ dK \\ dq \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix}$$

Provided $\begin{bmatrix} f_Y^{IS} & f_i^{IS} \\ f_Y^{LM} & f_i^{LM} \end{bmatrix}$ is invertible (which it will be if the conditions for the implicit function theorem to hold are satisfied), this can be written as

$$\begin{bmatrix} dY \\ di \end{bmatrix} =$$

$$-\begin{bmatrix} f_Y^{IS} & f_i^{IS} \\ f_Y^{LM} & f_i^{LM} \end{bmatrix}^{-1} \begin{bmatrix} f_P^{IS} & f_M^{IS} & f_B^{IS} & f_K^{IS} & f_q^{IS} & f_G^{IS} & f_\tau^{IS} & f_{\bar{T}}^{IS} & f_{\hat{\pi}}^{IS} \\ f_P^{LM} & f_M^{LM} & f_B^{LM} & f_K^{LM} & f_q^{LM} & f_G^{LM} & f_\tau^{LM} & f_{\bar{T}}^{LM} & f_{\hat{\pi}}^{LM} \end{bmatrix} \begin{bmatrix} dP \\ dM \\ dB \\ dK \\ dq \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix}$$

For the IS-LM model of equations 2.12 and 2.13, this exercise results in the following:

$$\begin{bmatrix} (1-\tau)c_{Y_d}+\iota_Y-1 & c_r+\iota_r \\ h_Y & h_i \end{bmatrix}\begin{bmatrix} dY \\ di \end{bmatrix} =$$

$$\begin{bmatrix} c_a\left(\frac{M+B}{P^2}\right) & -\frac{c_a}{P} & -\frac{c_a}{P} & -(qc_a+\iota_K) & -c_aK & -1 & Yc_{Y_d} & c_{Y_d} & c_r+\iota_r \\ -\left(\frac{M(1-h_a)-Bh_a}{P^2}\right) & \frac{1-h_a}{P} & -\frac{h_a}{P} & -qh_a & -Kh_a & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} dP \\ dM \\ dB \\ dK \\ dq \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix}$$

or

$$\begin{bmatrix} dY \\ di \end{bmatrix} = \begin{bmatrix} (1-\tau)c_{Y_d}+\iota_Y-1 & c_r+\iota_r \\ h_Y & h_i \end{bmatrix}^{-1}$$

$$\begin{bmatrix} c_a\left(\frac{M+B}{P^2}\right) & -\frac{c_a}{P} & -\frac{c_a}{P} & -(qc_a+\iota_K) & -c_aK & -1 & Yc_{Y_d} & c_{Y_d} & c_r+\iota_r \\ -\left(\frac{M(1-h_a)-Bh_a}{P^2}\right) & \frac{1-h_a}{P} & -\frac{h_a}{P} & -qh_a & -Kh_a & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} dP \\ dM \\ dB \\ dK \\ dq \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix}$$

The deteminant of the matrix $\begin{bmatrix} (1-\tau)c_{Y_d}+\iota_Y-1 & c_r+\iota_r \\ h_Y & h_i \end{bmatrix}$ is denoted D, that is,

$$D = ((1-\tau)c_{Y_d}+\iota_Y-1)\,h_i - h_Y(c_r+\iota_r) \tag{2.14}$$

We assume that the total marginal propensity to spend out of current income is less than one, that is, $(1-\tau)c_{Y_d}+\iota_Y-1<0$. It follows that $D>0$.

We also assume that a higher price level reduces the supply of real money balances by more than it reduces (through the wealth effect in money demand), the demand for real money balances, that is, $M(1-h_a)-Bh_a>0$.

2.1.1 For Future Reference: Budget Identities and Balance Sheet Constraints

Expectations and asset prices will be endogenised in later chapters. The behaviour over time of the predetermined asset stocks is, however, already implicit in the short-run equilibrium configuration developed thus far. For completeness these asset dynamics are made explicit here, even though no further use will be made of it in this chapter.

Each sector has a budget identity, specifying comprehensively all its uses and sources of funds. Each sector also has a financial balance sheet. Portfolio reshuffles at a point in time are constrained by the balance sheet constraint or identity.

Capital Stock Dynamics

Depreciation on the private capital stock is δK, where δ is the proportional depreciation rate of the private capital stock and K is the private capital stock. Depreciation on the public sector capital stock is $\delta^g K^g$, where δ^g is the proportional depreciation rate of the government capital stock and K^g is the government capital stock. Net changes in the physical capital stocks are an increasing, concave function of gross investment, I or I^g, minus depreciation:

$$\begin{aligned}\dot{K} &= \varphi(I)-\delta K\\ \varphi' &> 0\\ \varphi'(0) &\leq 1\\ \varphi'' &\leq 0\\ I &\geq 0\end{aligned}$$

and

$$\begin{aligned}
\dot{K}^g &= \varphi^g(I^g) - \delta^g K^g \\
\varphi^{g\prime} &> 0 \\
\varphi^{g\prime}(0) &\leq 1 \\
\varphi^{g\prime\prime} &\leq 0 \\
I^g &\geq 0
\end{aligned}$$

The Household Sector Budget Identity

Let S^h be the nominal value of gross household saving. Saving is disposable income minus consumption. Households are assumed to hold all base money and all government debt. All corporate earnings are paid out as dividends to households, so household factor income is total net national product or national income, Y, which is GDP, Q minus depreciation, $\delta K + \delta^g K^g$. All taxes are paid by households.

$$\dot{M} + \dot{B} + Pq\dot{K} \equiv S^h \equiv P(Y + i\frac{B}{P} - T - C) \tag{2.15}$$

The Household Sector Balance Sheet

Household portfolio reallocations at a point in time are constrained by:

$$dM + dB + PqdK \equiv 0 \tag{2.16}$$

The Enterprise Sector Budget Identity

Enterprises pay out all revenues from sales (assumed to equal production) to households. All corporate net investment is financed by issuing equity. S^e is the nominal value of gross enterprise saving (which equals depreciation of the private capital stock) and I is gross private capital formation.

$$P(I - \delta K - q\dot{K}) \equiv S^e - P\delta K \equiv 0 \tag{2.17}$$

The Government Sector Budget Identity

Total exhaustive government spending, G, is the sum of government consumption, C^g, and government investment, I^g, that is,

$$G = C^g + I^g$$

The nominal value of gross government saving is denoted S^g. Assume for simplicity that government capital does not yield any cash income to the government, and that the government capital stock stays permanently in the public sector. Privatisations and related issues are discussed briefly in chapter 9.

$$P(I^G - \delta^g K^g) - \dot{M} - \dot{B} \equiv S^g - P\delta^g K^g \equiv P(T - i\frac{B}{P} - C^g - \delta^g K^g) \quad (2.18)$$

The Government Sector Balance Sheet Constraint

Since we assume that government capital is not bought or sold, government portfolio reallocations (open market operations) at a point in time must be subject to the following balance sheet constraint or identity:

$$dM + dB \equiv 0 \quad (2.19)$$

Some National Account Identities

You can verify that the national accounts identities hold in this model. Let S be total gross national saving and $S^p \equiv S^h + S^e$ total gross private saving

$$S \equiv S^h + S^e + S^g \equiv P(I + I^g)$$

Total national gross saving equals total gross domestic capital formation (remember this is a closed economy).

$$S^p - I + S^g - I^g \equiv 0$$

The sum of all the sectoral financial surpluses or budget surpluses equals zero.

And of course,

$$C + C^g + I + I^g \equiv Y + \delta K + \delta^g K^g$$

The sectoral budget identities are not *constraints,* as there is nothing thus far to limit private or public sector borrowing. In later chapters we shall consider the intertemporal budget constraints for the private and public sectors that can be derived from the single-period or instantaneous budget identities

once a further constraint is added on the permissible behaviour of private and public financial debt. Often this takes the form of a non-negativity constraint on the terminal value of net financial assets (in the finite horizon case) or on the present value of terminal net financial assets (in the infinite horizon case). In Keynesian models of consumption, these rather weak intertemporal budget constraints or solvency constraints are often complemented with tighter restrictions on private financial wealth, such as $A(t) \geq 0$ for all t. This constraint, that household financial wealth is non-negative, would be appropriate if human wealth (the present value of future after-tax labour income) were not collateralisable.

2.1.2 The IS and LM Curves

The familiar IS curve is the locus of interest rate and output combinations that satisfy 2.10, the goods market equilibrium condition. Its slope in i, Y space is given by

$$\left(\frac{di}{dY}\right)_{IS} = \frac{1 - ((1-\tau)c_{Y_d} + \iota_Y)}{c_r + \iota_r} < 0 \tag{2.20}$$

It will be vertical if $c_r + \iota_r = 0$.

A change in any of the following will shift the IS curve in i, Y space: P, M, B, K, q, G, τ, $\bar{T}$, $\hat{\pi}$.

The equally familiar LM curve is the locus of interest rate and output combinations that satisfy 2.11, the monetary equilibrium condition. Its slope in i, Y space is given by

$$\left(\frac{di}{dY}\right)_{LM} = \frac{-h_Y}{h_i} \geq 0 \tag{2.21}$$

It will be vertical if $h_i = 0$, the interest-insensitive demand for money case or the constant income velocity of circulation of money case. It will be horizontal if $h_i = -\infty$, the Keynesian liquidity trap with its infinitely interest-sensitive demand for money. Japan is sometimes argued to be in that situation today. A change in any of the following will shift the LM curve in i, Y space: P, M, B, K, q.

2.1.3 Money Demand and Velocity

All statements about money demand can be rephrased as statements about the income velocity of circulation of money. The income velocity of circulation of base money, V is defined as follows:

$$V \equiv \frac{PY}{M}$$

From our money demand function, it follows that,

$$V = \frac{Y}{h(i, Y, a)}$$

A vertical LM curve, $h_i = 0$, also means that velocity is independent of the interest rate. For velocity to be constant, it is also necessary that there be no wealth effect on money demand: $h_a = 0$.

2.1.4 The Aggregate Demand Multipliers

The various reduced-form derivatives or aggregate demand multipliers are given by:

The 'Aggregate Demand Curve'

$$\frac{dY}{dP} = \frac{c_a\left(\frac{M+B}{P^2}\right)h_i + \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)(c_r + \iota_r)}{D} \leq 0 \tag{2.22}$$

The reciprocal of 2.22 gives the slope of the aggregate demand curve, the locus of IS-LM equilibrium levels of output for different values of the price level, that is

$$\frac{dP}{dY} = \frac{((1-\tau)c_{Y_d} + \iota_Y - 1)\,h_i - h_Y(c_r + \iota_r)}{c_a\left(\frac{M+B}{P^2}\right)h_i + \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)(c_r + \iota_r)} < 0 \tag{2.23}$$

.

Note that this will be vertical if $c_r + \iota_r = 0$ and $c_a = 0$. This is the case where the demand for goods and services is completely insensitive to the interest rate (a vertical IS curve in i, Y space) and there is no wealth effect or Pigou effect on consumption demand.

The aggregate demand curve will also be vertical (check D) if $h_i = -\infty$ and $c_a = 0$. This is the case of an infinitely interest-sensitive money demand function (the Keynesian liquidity trap, or a horizontal LM curve in i, Y space) and no wealth effect in the consumption function.

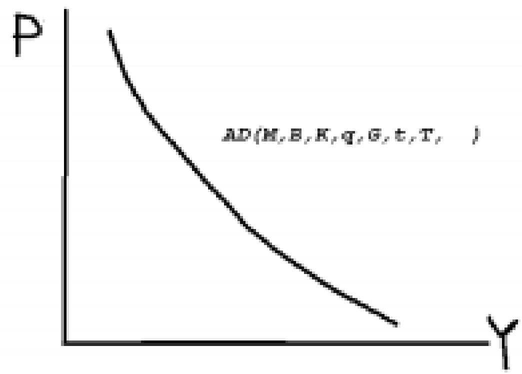

Figure 2.1: Aggregate Demand Curve; Normal Case

$$\frac{di}{dP} = \frac{[1 - ((1-\tau)c_{Y_d} + \iota_Y)]\left(\frac{M(1-h_a)-Bh_a}{P^2}\right) - h_Y c_a \left(\frac{M+B}{P^2}\right)}{D} \lesseqgtr 0 \tag{2.24}$$

A Helicopter Drop of Money

$$\frac{dY}{dM} = \frac{\frac{-c_a}{P}h_i - (c_r + \iota_r)\frac{(1-h_a)}{P}}{D} \geq 0 \tag{2.25}$$

The aggregate demand curve shifts to the right in P, Y space. It will not shift if $c_r + \iota_r = 0$ and $c_a = 0$. It will not shift either if $h_i = -\infty$ and $c_a = 0$. These are the same conditions that would make the aggregate demand curve vertical in PY space.

$$\frac{di}{dM} = \frac{[(1-\tau)c_{Y_d} + \iota_Y - 1]\left(\frac{(1-h_a)}{P}\right) + h_Y \frac{c_a}{P}}{D} \lesseqgtr 0 \tag{2.26}$$

Except for the two special cases resulting in a vertical AD curve that cannot be shifted by expansionary monetary policy, the IS curve shifts to the right (because of the wealth effect) and the LM curve shifts to the right.

A Helicopter Drop of Bonds

$$\frac{dY}{dB} = \frac{\frac{-c_a}{P}h_i + (c_r + \iota_r)\frac{h_a}{P}}{D} \lesseqgtr 0 \tag{2.27}$$

$$\frac{di}{dB} = \frac{\left[1 - ((1-\tau)c_{Y_d} + I_Y)\right]\frac{h_a}{P} + h_Y\frac{c_a}{P}}{D} > 0 \tag{2.28}$$

The IS curve shifts to the right (because of the wealth effect on consumption demand) and the LM curve shifts to the left (because of the wealth effect on money demand). Note that for government bonds to have a net wealth effect, there must be a violation of debt neutrality or Ricardian equivalence.

An Open Market Purchase of Bonds

$$\left(\frac{dY}{dM}\right)_{dM+dB=0} = \frac{\frac{-(c_r+\iota_r)}{P}}{D} \geq 0 \tag{2.29}$$

$$\left(\frac{di}{dM}\right)_{dM+dB=0} = \frac{\frac{\left((1-\tau)c_{Y_d}+\iota_Y-1\right)}{P}}{D} \leq 0 \tag{2.30}$$

There is no financial wealth effect. Only the LM curve shifts (to the right).

An Increase in the Capital Stock

$$\frac{dY}{dK} = \frac{-(qc_a + \iota_K)h_i + (c_r + \iota_r)qh_a}{D} \lesseqgtr 0 \tag{2.31}$$

$$\frac{di}{dK} = \frac{\left[1 - ((1-\tau)c_{Y_d} + \iota_Y)\right]qh_a + h_Y(qc_a + \iota_K)}{D} \lesseqgtr 0 \tag{2.32}$$

The LM curve shifts to the left (because of the wealth effect on money demand). The effect on the IS curve is ambiguous..

An Increase in the Stock Market Index

$$\frac{dY}{dq} = \frac{-Kc_a h_i + (c_r + \iota_r)Kh_a}{D} \lesseqgtr 0 \tag{2.33}$$

$$\frac{di}{dq} = \frac{\left[1 - ((1-\tau)c_{Y_d} + \iota_Y)\right] Kh_a + h_Y Kc_a}{D} > 0 \tag{2.34}$$

The IS curve shifts to the right because of the wealth effect on consumption. The LM curve shifts to the left because of the wealth effect on money demand.

An Increase in Public Spending on Goods and Services

$$\frac{dY}{dG} = \frac{-h_i}{D} \geq 0 \tag{2.35}$$

$$\frac{di}{dG} = \frac{h_Y}{D} > 0 \tag{2.36}$$

The IS curve shifts to the right. Output increases at a given price level unless the LM curve is vertical.

An Increase in the 'Marginal Propensity to Tax'

$$\frac{dY}{d\tau} = \frac{Yc_{Y_d}h_i}{D} \leq 0 \tag{2.37}$$

$$\frac{di}{d\tau} = \frac{-Yc_{Y_d}h_Y}{D} < 0 \tag{2.38}$$

The IS curve shifts to the left. Output falls unless the LM curve is vertical.

An Increase in Income-Independent Taxes

$$\frac{dY}{d\bar{T}} = \frac{(1-\tau)c_{Y_d}h_i}{D} \leq 0 \tag{2.39}$$

$$\frac{di}{d\bar{T}} = \frac{-(1-\tau)c_{Y_d}h_Y}{D} < 0 \tag{2.40}$$

The IS curve shifts to the left. Output falls unless the LM curve is vertical. Note that an equal increase in G and in $\bar{T}$ will shift the IS curve to the right. This is the balanced budget multiplier. The public spending multiplier exceeds (in magnitude) the income-independent tax multiplier.

Exercise 2.1.1 *We augment the fixed-price version of the AD model with a balanced budget condition.*

$$c\left(Y(1-\tau) - \bar{T}, i - \hat{\pi}, \frac{M+B}{P} + qK\right) + \iota(i - \hat{\pi}, Y, K) + G = Y$$

$$h(i, Y, \frac{M+B}{P} + qK) = \frac{M}{P}$$

$$G = \bar{T} + \tau Y$$

Find the effects on output and the interest rate of an increase in public spending (a) when the income-independent tax $\bar{T}$ varies endogenously to maintain budget balance and (b) when the marginal propensity to tax τ varies endogenously to maintain budget balance. Comment on your findings.

An increase in the expected rate of inflation

$$\frac{dY}{d\hat{\pi}} = \frac{(c_r + \iota_r)h_i}{D} \geq 0 \tag{2.41}$$

$$1 \geq \frac{di}{d\hat{\pi}} = \frac{-(c_r + \iota_r)h_Y}{D} \geq 0 \tag{2.42}$$

The IS curve shifts up and to the right (in i, Y space). The magnitude of the vertical shift of the IS curve is less than the increase in the expected rate of inflation unless the LM curve is vertical.

It follows that

$$0 \geq \frac{dr}{d\hat{\pi}} = \frac{-(c_r + \iota_r)h_Y}{D} - 1 \geq -1$$

The proposition that the real interest is independent of the (expected) rate of inflation is due to Irving Fisher.

Exercise 2.1.2 *Do a graphical analysis of the effect of an increase in the expected rate of inflation on output and the real interest rate, using an diagram with the real interest rate, r, on the vertical axis and output on the horizontal axis.*

2.1.5 Financial 'Crowding Out'

The fixed price model assumes that there are no capacity constraints on output. There is therefore no real resource constraint that limits the effect of an increase in public spending on output.

At a fixed interest rate, the increase in output associated with an increase in public spending on goods and services would be:

$$\left(\frac{dY}{dG}\right)_{i=\bar{\imath}} = \frac{1}{1-((1-\tau)c_{Y_d}+\iota_Y)} > 0 \tag{2.43}$$

This is, of course, the simple public spending multiplier from the 'Keynesian Cross'. The change in output given in 2.43 corresponds to the horizontal shift of the IS curve in i, Y space. When income expands, the transactions demand for money increases. In order to maintain monetary equilibrium, the interest rate has to rise, crowding out interest-sensitive private spending. This is a form of 'financial crowding out' due to non-accommodating monetary policy.

2.1.6 Nominal Interest Rate Pegging

In practice central banks do not use the money stock as an instrument. Broad monetary aggregates, which consist mainly of liabilities of the banking system and of other deposit-taking institutions, cannot be directly controlled by the monetary authorities. Even narrow money, M_0, the monetary base or high-powered money (currency and commercial bank deposits with the central bank), which in principle could be controlled exactly, are not, in practice used as an instrument of monetary policy. Central banks use open market operations and variations in reserve requirements to influence money market conditions and other financial markets, but the main instrument is a short-term nominal interest rate. In the UK, a 2-week repo rate is the operational instrument.

In the IS-LM model with a fixed price level, we can analyse this representation of monetary policy by treating i as the policy instrument. The nominal money stock then becomes an endogenous variable. Consider again the IS-LM model below.

$$c\left(Y(1-\tau)-\bar{T}, i-\hat{\pi}, \frac{M+B}{P}+qK\right)+\iota(i-\hat{\pi}, Y, K)+G=Y$$

$$h(i, Y, \frac{M+B}{P}+qK)=\frac{M}{P}$$

With $i=\bar{\imath}$, the LM curve becomes horizontal in i, Y space. We are effectively in the Keynesian liquidity trap, not because of an infinitely interest-sensitive demand for money but because of an infinitely interest-sensitive supply of money. Note that, while M is endogenous, in the short run $L \equiv M+B$, the total stock of financial liabilities of the government, is predetermined. The composition of $M+B$ between M and B is endogenous.

The reduced form aggregate demand derivatives or multipliers can be found from:

$$\begin{bmatrix} (1-\tau)c_{Y_d}+\iota_Y-1 & 0 \\ h_Y & -\frac{1}{P} \end{bmatrix}\begin{bmatrix} dY \\ dM \end{bmatrix}=$$

$$\begin{bmatrix} c_a\frac{L}{P^2} & -(c_r+\iota_r) & -\frac{c_a}{P} & -(qc_a+\iota_K) & -c_aK & -1 & Yc_{Y_d} & c_{Y_d} & c_r+\iota_r \\ \frac{Lh_a-M}{P^2} & -h_i & -\frac{h_a}{P} & -qh_a & -Kh_a & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} dP \\ di \\ dL \\ dK \\ dq \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix}$$

For instance, the price level, nominal interest rate and government spending multipliers are given by

$$\Delta=\frac{1-(1-\tau)c_{Y_d}-\iota_Y}{P}>0$$

$$\frac{dY}{dP}=\frac{-c_aL}{\Delta P^3}<0$$

$$\frac{dM}{dP}=\frac{((1-\tau)c_{Y_d}+\iota_Y-1)\left(\frac{Lh_a-M}{P^2}\right)-h_Yc_a\frac{L}{P^2}}{\Delta} \lesseqgtr 0$$

The aggregate demand curve will be downward-sloping if and only if there is a wealth effect on consumption and if there is a positive stock of government financial liabilities outstanding. With M exogenous, the nominal interest rate would be higher if the price level were higher. This would reduce demand through the interest-sensitive components of private investment and consumption. This channel is absent when the nominal interest rate is pegged. The effect on the nominal money stock is ambiguous.. Ceteris paribus, a higher price level lowers the real money stock corresponding to any given nominal money stock. Output, and the transactions demand for money, also fall, however, and there is also a negative wealth effect of a higher price level on the demand for money.

$$\frac{dY}{di} = \frac{c_r + \iota_r}{P\Delta} < 0$$

$$\frac{dM}{di} = \frac{h_i\left((1-\tau)c_{Y_d} + \iota_Y - 1\right) + h_Y\left(c_r + \iota_r\right)}{\Delta} < 0$$

The horizontal LM curve shifts up. Output falls and with it the demand for money and the nominal money stock.

$$\frac{dY}{dG} = \frac{1}{P\Delta} > 0$$

$$\frac{dM}{dG} = \frac{h_Y}{\Delta} > 0$$

The government spending multiplier in this case is the 'Keynesian cross' multiplier, since the nominal interest rate is pegged.

2.1.7 The Bank Credit Channel

There is a long-standing tradition, going back at least to Tobin, Brainard and Brunner and Meltzer, that views the monetary transmission mechanism as working not only through the monetary equilibrium condition and the short nominal rate of interest, but also through the loan rate, the rate of interest charged by banks for loans to households and businesses. There has been a revival of this view in recent years, associated with the work of Bernanke, Gertler, Blinder and others.

The starting point of the analysis is a theory of capital market distortions or imperfections that make the various sources of funds for enterprises imperfect substitutes for each other. Internal funds (undistributed profits) are viewed as having the lowest cost, followed by bank loans, corporate bond issuance and equity issues. The main reason for the imperfect substitutability among these different sources of funds are asymmetric information and costly state verification. Incomplete contract models based on bounded rationality can have similar implications. Limited liability and differences in the tax treatment of the returns to different financial instruments can also play a role. So can the large fixed cost of accessing the equity or corporate bond markets. This leads to problems of adverse selection and moral hazard. Ownership and control are exercised by distinct agents. The corporation ceases to be a veil. Corporate governance issues matter and the interests of entrepreneurs, providers of risk capital, banks and other lenders are no longer automatically aligned.

The bank credit channel focuses on the distinction between the interest rate on government bonds, i, which, give or take a risk premium, is also the corporate bond rate. In addition, corporations have access to bank lending at a rate i_l. For small and medium-sized enterprises, bank lending may indeed be the only form of external finance.

The first modification of the model is to make investment, and possibly household consumption as well, a decreasing function of the real loan rate, $r_l \equiv i_l - \hat{\pi}$, as well as of the real interest rate, that is

$$\begin{aligned} I &= \iota(r, r_l, Y, K) \\ \iota_r &< 0; \iota_{r_l} < 0; \iota_Y > 0; \iota_K < 0 \end{aligned}$$

$$\begin{aligned} C &= c(r, r_l, Y_d, a) \\ c_r &\leq 0; c_{r_l} \leq 0; 0 < c_{Y_d} < 1; c_a \geq 0 \end{aligned}$$

The second modification is to introduce an equilibrium relationship for the demand for loans, Λ^d, and supply of loans, Λ^s.

The demand for loans by non-bank enterprises and households depends negatively on the real loan rate and positively on the real interest rate. When both the real loan rate and the real interest rate increase by the same amount, the demand for loans falls. The demand for loans also depends negatively on the cost of equity capital, proxied by q.

$$\begin{gathered}\Lambda^d = \lambda^d(r, r_l, q)\\ \lambda_r^d > 0; \lambda_{r_l}^d < 0; \lambda_r^d + \lambda_{r_l}^d < 0; \lambda_q^d < 0\end{gathered} \tag{2.44}$$

The supply of loans by banks is part of the asset side of the banks' balance sheet. On the liability side are deposits, on the asset side loans, government bonds and reserves, which are part of the monetary base M. The supply of loans increases with the real loan rate and falls with the real bond rate. When both rates increase by the same amount, the supply of loans increases. The supply of loans also increases, for any given loan rate and bond rate, with the value of private (strictly speaking private non-bank) financial wealth. Financial wealth can be used as collateral to secure loans. This mitigates the adverse selection and moral hazard problems restricting the supply of loans. Finally, a larger value of the real monetary base increases the supply of loans. The simplest way to achieve this result is to assume that bank reserves are a fixed fraction of the stock of bank deposits, but a less rigid approach to bank reserves can generate the same result.

$$\begin{gathered}\Lambda^s = \lambda^s(r, r_l, a, \tfrac{M}{P})\\ \lambda_r^s < 0; \lambda_{r_l}^s > 0; \lambda_r^s + \lambda_{r_l}^s > 0; \lambda_a^s > 0; \lambda_{M/P}^s > 0\end{gathered} \tag{2.45}$$

$$\Lambda^s = \Lambda^d \tag{2.46}$$

Our simple short-run equilibrium model now becomes

$$c\left(Y(1-\tau) - \bar{T}, i - \hat{\pi}, i_l - \hat{\pi}, \frac{M+B}{P} + qK\right) + \iota(i - \hat{\pi}, i_l - \hat{\pi}, Y, K) + G = Y \tag{2.47}$$

$$h(i, Y, \frac{M+B}{P} + qK) = \frac{M}{P} \tag{2.48}$$

$$\lambda^s(i - \hat{\pi}, i_l - \hat{\pi}, \frac{M+B}{P} + qK, \frac{M}{P}) = \lambda^d(i - \hat{\pi}, i_l - \hat{\pi}, q) \tag{2.49}$$

In the fixed price version of the model, we now determine the 3 equilibrium variables i, i_l, and Y.

While slightly more involved, the transmission channels of monetary policy are very similar. An increase in M will reduce both i and i_l and raise Y.

2.2 The Flexible Price Case

The supply side of the model is given by a neoclassical production function with capital and labour, L as its two inputs. Output also depends on the level of technology, indexed by Θ

$$Y = F(K, L; \Theta) \tag{2.50}$$

It has constant returns to scale,

$$\begin{aligned} F(\lambda K, \lambda L; \Theta) = & \quad \lambda F(K, L; \Theta) \\ \lambda \geq & \quad 0 \end{aligned}$$

positive but diminishing marginal products of labour and capital, and is strictly concave.

$$\begin{aligned} F_L &> 0 \\ F_K &> 0 \\ F_{LL} &< 0 \\ F_{KK} &< 0 \end{aligned}$$

Since Θ stands for the level of efficiency, it follows (by definition) that

$$F_\Theta > 0$$

Note that, with constant returns to scale, the production function can be written in 'intensive' form:

$$y = f(k, \Theta)$$

where $k \equiv \frac{K}{L}$ and $y \equiv \frac{Y}{L}$.

We also assume that the production function satisfies the Inada conditions:

$$\begin{aligned} f(0, \Theta) &= 0 \\ \lim_{k \to 0} f_k(k, \Theta) &= \infty \\ \lim_{k \to \infty} f_k(k, \Theta) &= 0 \end{aligned}$$

An example of a production function satisfying all this is the Cobb-Douglas:

$$\begin{aligned} Y &= \Theta K^{\alpha} L^{1-\alpha} \\ 0 &< \alpha < 1; \Theta > 0 \end{aligned}$$

Another useful production function that satisfies all except the Inada conditions is the Constant Elasticity of Substitution:

$$\begin{aligned} Y &= \Theta \left(aK^{-\rho} + bL^{-\rho} \right)^{-\frac{1}{\rho}} \\ \Theta &> 0; a, b > 0; \rho \geq -1 \end{aligned}$$

Exercise 2.2.1 *Check that the limit of the CES production function as $\rho \rightarrow 0$ is the Cobb-Douglas production function. (Hint: use l'Hôspital's rule).*

Another useful constant returns to scale production function is the fixed coefficient, Leontieff or input-output production function:

$$\begin{aligned} Y &= \Theta \min \left\{ \frac{K}{a}, \frac{L}{b} \right\} \\ \Theta, a, b &> 0 \end{aligned}$$

A final, not terribly useful, constant returns to scale production function is the linear production function, with its infinite elasticity of substitution:

$$\begin{aligned} Y &= aK + bL \\ a, b &> 0 \end{aligned}$$

In the short run, competitive, profit-maximising producers only choose their variable labour input. The capital stock is predetermined. There is a tax at a constant proportional rate τ_w on the wage bill (a proportional payroll tax, social security tax or national insurance contribution by the employer). The money wage is denoted W. There also is an indirect tax at a constant proportional rate τ_i. If P is the price level at market prices, the producer only gets the price level at factor cost, $\frac{P}{1+\tau_i}$. Labour demand is denoted L^d and labour supply L^s.

Profits, Π, are given by

$$\Pi = \frac{P}{1+\tau_i}Y - W(1+\tau_w)L \tag{2.51}$$

Substituting the production function into 2.51 for Y, we get the profit-maximising first-order condition

$$\frac{W(1+\tau_w)(1+\tau_i)}{P} = F_L(L^d, K, \Theta) \tag{2.52}$$

The marginal product of labour equals the marginal cost of labour. The worker's take home real wage, $\frac{W}{P}$, is not the only component of the marginal cost of labour. Payroll taxes and indirect taxes drive a wedge between real take-home pay and the real marginal cost to the employer of employing an additional worker.

We assume that the supply of labour is a non-decreasing function of the worker's after-tax wage. There is a constant proportional income tax rate, τ_y applied to all wage income. This drives a further wedge between the after-tax wage of the worker and the marginal cost of labour to the employer. Note that the labour supply relationship is written in terms of the supply price of labour $\frac{\frac{W}{1+\tau_y}}{P}$, which is non-decreasing in the amount of labour supplied.

$$\begin{aligned} \frac{\frac{W}{1+\tau_y}}{P} &= \ell(L^s) \\ \ell' &\geq 0 \end{aligned} \tag{2.53}$$

We assume that the labour market clears. The three key assumptions for the labour market are therefore that (a) it is competitive, (b) both demand and supply depend on real variables only (there is no money illusion of any kind) and (c), the real wage is perfectly flexible and clears the labour market instantaneously.

$$L^s = L^d = L \tag{2.54}$$

Labour market equilibrium is therefore given by:

$$F_L(K, L, \Theta)\frac{1}{(1+\tau_w)(1+\tau_i)(1+\tau_y)} = \ell(L) \tag{2.55}$$

The key point to note is that, with the capital stock and technology given, the equilibrium level of employment. depends only on the three (distortionary) tax rates. It is independent of anything nominal. It is also independent of public spending, G, and of the aggregate demand effects of taxes. Only the 'tax wedges' τ_w, τ_i and τ_y affect the equilibrium level of employment, through conventional microeconomic incentive effects or 'supply-side' effects. Given K, Θ, τ_w, τ_i and τ_y, the real wage and the level of real output will also be independent of the 'demand side' of the economy. The aggregate supply curve will therefore be vertical in P, Y space.

Equilibrium employment is reduced by increases in any of the three tax wedges. The vertical AS (aggregate supply) curve in P, Y space shifts to the left.

We can solve equilibrium employment implicitly from 2.55.

$$L = l(\Theta, K, \tau_w, \tau_i, \tau_y)$$

$$l_\Theta = \frac{F_{L\Theta}}{(1+\tau_w)(1+\tau_i)(1+\tau_y)\Omega} \lesseqgtr 0$$

$$l_K = \frac{F_{LK}}{(1+\tau_w)(1+\tau_i)(1+\tau_y)\Omega} \geq 0$$

$$l_{\tau_w} = \frac{-\ell(L)}{(1+\tau_w)\Omega} < 0$$

$$l_{\tau_i} = \frac{-\ell(L)}{(1+\tau_i)\Omega} < 0$$

$$l_{\tau_y} = \frac{-\ell(L)}{(1+\tau_y)\Omega} < 0$$

$$\Omega = \ell' - \frac{F_{LL}}{(1+\tau_w)(1+\tau_i)(1+\tau_y)} > 0$$

The effect of technical change on equilibrium employment depends on what technical change does to the marginal product of labour, that is, on the sign and magnitude of $F_{L\Theta}$. If technical change is biased against labour (labour-saving), then $F_{L\Theta} < 0$. The effect of a larger capital stock on equilibrium employment depends on whether labour and capital are substitutes ($F_{LK} < 0$) or complements ($F_{LK} > 0$). With a two-input constant returns to scale, strictly concave production function, with positive but diminishing marginal products of both inputs, the two inputs cannot be substitutes, so $F_{LK} \geq 0$. Note that this does not hold if there are more than two inputs (say land, labour and capital or energy, labour and capital).

Output supply can, using 2.55 and the production function be written as

$$Y = F\left(K, l(\Theta, K, \tau_w, \tau_i, \tau_y), \Theta\right) \tag{2.56}$$

For the flexible price case, we can substitute the full-employment level of output (from 2.56) into the IS and LM equilibrium conditions. This yields

$$\begin{aligned} & c\left(F\left(K, l(\Theta, K, \tau_w, \tau_i, \tau_y), \Theta\right)(1-\tau) - \bar{T}, i - \hat{\pi}, \frac{M+B}{P} + qK\right) \\ & + \iota\left[i - \hat{\pi}, F\left(K, l(\Theta, K, \tau_w, \tau_i, \tau_y), \Theta\right), K\right] + G \\ = \; & F\left(K, l(\Theta, K, \tau_w, \tau_i, \tau_y), \Theta\right) \end{aligned} \tag{2.57}$$

$$h(i, F\left(K, l(\Theta, K, \tau_w, \tau_i, \tau_y), \Theta\right), \frac{M+B}{P} + qK) = \frac{M}{P} \tag{2.58}$$

We can now do the comparative statics of changes in the exogenous variables M, B, K, q, G , τ, $\bar{T}$, $\hat{\pi}, \tau_w, \tau_i, \tau_y$ and Θ on the two endogenous variables, P and i. The most useful space for drawing the IS-LM equilibrium schedules now is obviously i, P space, rather than i, Y space. Note that if we change τ or $\bar{T}$, while holding τ_w, τ_i and τ_y constant, we must be changing some tax that does affect the supply side through the labour market. Likewise, if we change τ_w, τ_i or τ_y while keeping τ and $\bar{T}$, there must be some unseen taxes in the background that vary in such a way as to neutralise any aggregate demand effects from changes in τ_w, τ_i or τ_y.

The slope of the IS curve in i, P space is given by

$$\left(\frac{di}{dP}\right)_{IS} = \frac{c_a\left(\frac{M+B}{P^2}\right)}{c_r + \iota_r} < 0 \tag{2.59}$$

The IS curve will be shifted by changes in $M, B, K, q, G, \tau, \bar{T}, \hat{\pi}, \tau_w, \tau_i, \tau_y$ and Θ.

The slope of the LM curve in i, P space is given by

$$\left(\frac{di}{dP}\right)_{LM} = \frac{-\left(\frac{M(1-h_a)-Bh_a}{P^2}\right)}{h_i} \geq 0 \tag{2.60}$$

The LM curve will be shifted by changes in $M, B, K, q, \tau_w, \tau_i, \tau_y$ and Θ.

The linearized structural model is given below:

$$\begin{bmatrix} -c_a\left(\frac{M+B}{P^2}\right) & c_r + \iota_r \\ \frac{M(1-h_a)-Bh_a}{P^2} & h_i \end{bmatrix} \begin{bmatrix} dP \\ di \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{c_a}{P} & -\frac{c_a}{P} & \mu(F_K + F_L l_K) - (qc_a + \iota_K) & -c_a K & -1 & Y c_{Y_d} & c_{Y_d} & c_r + \iota_r \\ \frac{1-h_a}{P} & -\frac{h_a}{P} & -(qh_a + h_Y(F_K + F_L l_K)) & -Kh_a & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dM \\ dB \\ dK \\ dq \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix}$$

$$+ \begin{bmatrix} \mu F_L l_{\tau_w} & \mu F_L l_{\tau_i} & \mu F_L l_{\tau_y} & \mu(F_\Theta + F_L l_\Theta) \\ -h_Y F_L l_{\tau_w} & -h_Y F_L l_{\tau_i} & -h_Y F_L l_{\tau_y} & -h_Y(F_\Theta + F_L l_\Theta) \end{bmatrix} \begin{bmatrix} d\tau_w \\ d\tau_i \\ d\tau_y \\ \Theta \end{bmatrix} \tag{2.61}$$

where

$$\mu \equiv 1 - ((1-\tau)c_{Y_d} + \iota_Y) > 0$$

The determinant of the matrix $\begin{bmatrix} -c_a\left(\frac{M+B}{P^2}\right) & c_r + \iota_r \\ \frac{M(1-h_a)-Bh_a}{P^2} & h_i \end{bmatrix}$ is again denoted D,that is,

$$D = -c_a\left(\frac{M+B}{P^2}\right)h_i - \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)(c_r + \iota_r) > 0$$

A Helicopter Drop of Money

$$\frac{dP}{dM} = \frac{\frac{-c_a}{P}h_i - (c_r + \iota_r)\frac{(1-h_a)}{P}}{D} \geq 0 \tag{2.62}$$

The aggregate demand curve shifts to the right in P, Y space. The aggregate supply curve does not shift. We can check for neutrality of money as follows:

$$\frac{MdP}{PdM} = \frac{c_a\frac{M}{P^2}h_i + \frac{(1-h_a)}{P^2}(c_r + \iota_r)}{c_a\left(\frac{M+B}{P^2}\right)h_i + \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)(c_r + \iota_r)} \geq 0 \tag{2.63}$$

It follows that money is neutral only if $B = 0$.

$$\frac{di}{dM} = \frac{-c_a\left(\frac{M+B}{P^2}\right)\left(\frac{(1-h_a)}{P}\right) + \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)\frac{c_a}{P}}{D} \leq 0 \tag{2.64}$$

If $B = 0$,then $\frac{di}{dM} = 0$

A Helicopter Drop of Bonds

$$\frac{dP}{dB} = \frac{\frac{-c_a}{P}h_i + (c_r + \iota_r)\frac{h_a}{P}}{D} \lesseqgtr 0 \tag{2.65}$$

$$\frac{di}{dB} = \frac{c_a\left(\frac{M+B}{P^2}\right)\frac{h_a}{P} + \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)\frac{c_a}{P}}{D} > 0 \tag{2.66}$$

The IS curve shifts to the right (because of the wealth effect on consumption demand) and the LM curve shifts to the left (because of the wealth effect on money demand). The shift of the aggregate demand curve is ambiguous. The AS curve does not shift.

An Open Market Purchase of Bonds

$$\left(\frac{dP}{dM}\right)_{dM+dB=0} = \frac{\frac{-(c_r+\iota_r)}{P}}{D} \geq 0 \tag{2.67}$$

$$\left(\frac{di}{dM}\right)_{dM+dB=0} = \frac{\frac{-c_a(M+B)}{P^3}}{D} \leq 0 \tag{2.68}$$

There is no financial wealth effect. Only the LM curve shifts (to the right). The AD curve shifts to the right. The AS curve does not shift.

An Equiproportional Helicopter Drop of Money and Bonds

$$\left(\frac{(M+B)dP}{Pd(M+B)}\right)_{\frac{dM}{M}=\frac{dB}{B}} = 1$$

$$\left(\frac{di}{d(M+B)}\right)_{\frac{dM}{M}=\frac{dB}{B}} = 0$$

An equiproportional increase in the nominal stocks of money AND bonds leads to an equal proportional increase in the price level. All nominal 'outside' assets are jointly neutral. The AD curve shifts to the right.

An Increase in the Capital Stock

$$\frac{dP}{dK} = \frac{\left(\mu(F_K + F_L l_K) - (qc_a + \iota_K)\right) h_i + (c_r + \iota_r)\left(qh_a + h_Y(F_K + F_L l_K)\right)}{D} \lesseqgtr 0 \tag{2.69}$$

$$\frac{di}{dK} = \frac{c_a\left(\frac{M+B}{P^2}\right)\left(qh_a + h_Y(F_K + F_L l_K)\right) - \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)\left(\mu(F_K + F_L l_K) - (qc_a + \iota_K)\right)}{D} \lesseqgtr 0 \tag{2.70}$$

The LM curve shifts to the left (because of the wealth effect on money demand). The effect on the IS curve is ambiguous. The AS curve shifts to the right.

An Increase in the Stock Market Index

$$\frac{dP}{dq} = \frac{-Kc_a h_i + (c_r + \iota_r)Kh_a}{D} \gtreqless 0 \tag{2.71}$$

$$\frac{di}{dq} = \frac{c_a\left(\frac{M+B}{P^2}\right)Kh_a + \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)Kc_a}{D} > 0 \tag{2.72}$$

The IS curve shifts to the right because of the wealth effect on consumption. The LM curve shifts to the left because of the wealth effect on money demand. The AS curve does not shift.

An Increase in Public Spending on Goods and Services

$$\frac{dP}{dG} = \frac{-h_i}{D} \geq 0 \tag{2.73}$$

$$\frac{di}{dG} = \frac{\left(\frac{M(1-h_a)-Bh_a}{P^2}\right)}{D} > 0 \tag{2.74}$$

The IS curve shifts to the right. Output increases at a given price level (the AD curve shifts to the right) unless the LM curve is vertical. The AS curve does not shift.

An Increase in the 'Marginal Propensity to Tax'

$$\frac{dP}{d\tau} = \frac{Yc_{Y_d}h_i}{D} \leq 0 \tag{2.75}$$

$$\frac{di}{d\tau} = \frac{-\left(\frac{M(1-h_a)-Bh_a}{P^2}\right)Yc_{Y_d}}{D} < 0 \tag{2.76}$$

The IS curve shifts to the left. Output falls at a given price level (the AD curve shifts to the left) unless the LM curve is vertical. AS does not shift.

An Increase In Income-Independent Taxes

$$\frac{dP}{d\bar{T}} = \frac{c_{Y_d}h_i}{D} \leq 0 \tag{2.77}$$

$$\frac{di}{d\bar{T}} = \frac{-\left(\frac{M(1-h_a)-Bh_a}{P^2}\right)c_{Y_d}}{D} < 0 \tag{2.78}$$

The IS curve shifts to the left. Output falls at a given price level (the AD curve shifts to the left) unless the LM curve is vertical.

An Increase In The Expected Rate of Inflation

$$\frac{dP}{d\hat{\pi}} = \frac{(c_r + \iota_r)h_i}{D} \geq 0 \tag{2.79}$$

$$1 \geq \frac{di}{d\hat{\pi}} = \frac{-\left(\frac{M(1-h_a)-Bh_a}{P^2}\right)(c_r + \iota_r)}{c_a\left(\frac{M+B}{P^2}\right)h_i + \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)(c_r + \iota_r)} \geq 0 \tag{2.80}$$

Note that $\frac{di}{d\hat{\pi}} = 1$ (or, equivalently, $\frac{dr}{d\hat{\pi}} = 0$) either if money demand is interest-inelastic ($h_i = -\infty$) or if there is no wealth effect on consumption ($c_a = 0$).

The Distortionary Tax Multipliers

$$\frac{dP}{d\tau_w} = \frac{F_L l_{\tau_w}\left(\mu h_i + h_Y(c_r + \iota_r)\right)}{D} > 0 \tag{2.81}$$

$$\frac{di}{d\tau_w} = \frac{F_L l_{\tau_w}\left[c_a\left(\frac{M+B}{P^2}\right)h_Y - \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)\mu\right]}{D} \lesseqgtr 0 \tag{2.82}$$

$$\frac{dP}{d\tau_i} = \frac{F_L l_{\tau_i}\left(\mu h_i + h_Y(c_r + \iota_r)\right)}{D} > 0 \tag{2.83}$$

$$\frac{di}{d\tau_i} = \frac{F_L l_{\tau_i}\left[c_a\left(\frac{M+B}{P^2}\right)h_Y - \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)\mu\right]}{D} \lesseqgtr 0 \tag{2.84}$$

$$\frac{dP}{d\tau_y} = \frac{F_L l_{\tau_y}\left(\mu h_i + h_Y(c_r + \iota_r)\right)}{D} > 0 \tag{2.85}$$

$$\frac{di}{d\tau_y} = \frac{F_L l_{\tau_y}\left[c_a\left(\frac{M+B}{P^2}\right)h_Y - \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)\mu\right]}{D} \lesseqgtr 0 \tag{2.86}$$

For all three tax wedges, the aggregate demand curve does not shift. The aggregate supply curve shifts to the left.

$$\begin{aligned}\frac{dP}{d\Theta} &= \frac{(F_\Theta + F_L l_\Theta)(\mu h_i + h_Y(c_r + \iota_r))}{D} < 0 \\ &\text{(probably)}\end{aligned} \tag{2.87}$$

$$\frac{di}{d\Theta} = \frac{(F_\Theta + F_L l_\Theta)\left[c_a\left(\frac{M+B}{P^2}\right)h_Y - \left(\frac{M(1-h_a)-Bh_a}{P^2}\right)\mu\right]}{D} \lesseqgtr 0 \tag{2.88}$$

The AS curve shifts to the right. The AD curve does not shift.
The real output multipliers are as follows:

$$\frac{dY}{dM} = \frac{dY}{dB} = \frac{dY}{dq} = \frac{dY}{dG} = \frac{dY}{d\tau} = \frac{dY}{d\bar{T}} = \frac{dY}{d\hat{\pi}} = 0$$

$$\frac{dY}{dK} = F_K + F_L l_K > 0$$

$$\frac{dY}{d\tau_w} = F_L l_{\tau_w} < 0$$

$$\frac{dY}{d\tau_i} = F_L l_{\tau_i} < 0$$

$$\frac{dY}{d\tau_y} = F_L l_{\tau_y} < 0$$

$$\begin{aligned}\frac{dY}{d\Theta} &= F_\Theta + F_L l_\Theta \lesseqgtr 0 \\ \text{(probably} &> 0)\end{aligned}$$

An interesting property of the model is that revenue-neutral changes in the mix of income tax, indirect tax and payroll tax have no effect on the general price level (or on anything else).

2.2.1 Some Alternative Classical Supply-Side Models

Intertemporal Substitution in Labour Supply

For simplicity, ignore real capital and technical change in what follows. The production function therefore becomes

$$\begin{aligned} Y &= F(L) \\ F' &> 0 \\ F'' &< 0 \\ F(0) &= 0 \end{aligned}$$

Distortionary taxes are also ignored, $\tau_w = \tau_i = \tau_y = 0$, and aggregate taxes net of transfers are independent of income, $\tau = 0$.

The intertemporal substitution theory of labour supply implies that current labour supply, L_t^s, is an increasing function of the current real wage, $\frac{W_t}{P_t}$ and a decreasing function of the present discounted value of next period's real wage, $\frac{(W_{t+1}/P_{t+1})^e}{1+r_{t,t+1}}$ (see e.g. Hall [17]).

$$\begin{aligned} L_t^s &= j\left(\frac{W_t}{P_t}, \frac{(W_{t+1}/P_{t+1})^e}{1+r_{t,t+1}}\right) \\ j_1 &> 0;\ j_2 < 0 \end{aligned} \tag{2.89}$$

This is consistent with there being no effect on the supply of labour from. a permanent increase in the real wage. This would require:

$$j_1 + \frac{j_2}{1+r} = 0$$

Suppressing the expected future real wage, we can rewrite 2.89 as

$$\begin{aligned} L^s &= j\left(\frac{W}{P}, r\right) \\ j_1 &> 0;\ j_2 > 0 \end{aligned}$$

Labour supply is therefore increasing in the real rate of interest.

Labour market equilibrium is now given by

$$F'(L) = \frac{W}{P} \tag{2.90}$$

and

$$L = j\left(\frac{W}{P}, r\right) \tag{2.91}$$

Equations 2.90 and 2.91 can be used to eliminate L and express the real wage as a decreasing function of the real interest rate:

$$\begin{aligned} \frac{W}{P} &= \psi(r) \\ \psi' &= \frac{F'' j_2}{1 - F'' j_1} < 0 \end{aligned}$$

We can now rewrite the IS-LM, AD-AS model as follows:

$$\begin{aligned} & c\left(F\left(j(\psi(i-\hat{\pi}), i-\hat{\pi}\right) - \bar{T}, i-\hat{\pi}, \frac{M+B}{P}\right) + \iota(i-\hat{\pi}, F\left(j(\psi(i-\hat{\pi}), i-\hat{\pi}\right) + G \\ = \; & F\left(j(\psi(i-\hat{\pi}), i-\hat{\pi}\right) \end{aligned} \tag{2.92}$$

$$h\left[i, F\left(j[\psi(i-\hat{\pi}), i-\hat{\pi}]\right), \frac{M+B}{P}\right] = \frac{M}{P} \tag{2.93}$$

The linearized structural model is given by

$$\begin{bmatrix} c_a\left(\frac{M+B}{P^2}\right) & (1-c_{Y_d}-\iota_Y)F'(j_1\psi'+j_2)-(c_r+\iota_r) \\ -\frac{M(1-h_a)-Bh_a}{P^2} & -\left(h_i+h_Y F'(j_1\psi'+j_2)\right) \end{bmatrix} \begin{bmatrix} dP \\ di \end{bmatrix} =$$

$$\begin{bmatrix} \frac{c_a}{P} & \frac{c_a}{P} & 1 & -c_{Y_d} & (1-c_{Y_d}-\iota_Y)F'(j_1\psi'+j_2)-(c_r+\iota_r) \\ \frac{h_a-1}{P} & \frac{h_a}{P} & 0 & 0 & -h_Y F'(j_1\psi'+j_2) \end{bmatrix} \begin{bmatrix} dM \\ dB \\ dG \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix} \tag{2.94}$$

Note that

$$j_1\psi' + j_2 = \frac{j_2}{1 - F''j_1} > 0$$

A higher real interest rate is therefore associated with higher equilibrium employment (from the supply side alone).

The LM curve is upward-sloping in i, P space if

$$\frac{M(1-h_a) - Bh_a}{P^2} > 0$$

and

$$h_i + h_Y F'(j_1\psi' + j_2) < 0 \tag{2.95}$$

The first of these two assumptions has been standard so far. The second requires that a higher interest rate reduces the demand for money, even though there now is a second effect of a higher rate of interest on money demand: a higher interest rate raises employment, output and the transactions demand for money.

The determinant of the matrix

$$\begin{bmatrix} c_a\left(\frac{M+B}{P^2}\right) & (1 - c_{Y_d} - \iota_Y)F'(j_1\psi' + j_2) - (c_r + \iota_r) \\ -\frac{M(1-h_a)-Bh_a}{P^2} & -\left(h_i + h_Y F'(j_1\psi' + j_2)\right) \end{bmatrix}$$

denoted D, is positive.

An increase in public spending has the following effect on the price level and the nominal (and real) interest rate:

$$\frac{dP}{dG} = \frac{-\left(h_i + h_Y F'(j_1\psi' + j_2)\right)}{D} > 0 \text{ if 2.95 is satisfied}$$

$$\frac{di}{dG} = \frac{\frac{M(1-h_a)-Bh_a}{P^2}}{D} > 0 \tag{2.96}$$

Higher government spending raises the nominal interest rate. With expected inflation given, the real interest rate increases also. Employment rises because of the intertemporal substitution effect on labour supply. Output increases. Note that it is the full employment level of output that increases: the vertical AS curve shifts to the right. Because of 2.95, the price level rises. While the aggregate supply curve shifts to the right, 2.95is sufficient to ensure that the shift of the AD curve dominates.

A Working Capital Model

Now consider the case where there is a one-period lag in the production process: the labour input in period t does not result in additional output until period $t+1$. The production function can now be written as

$$\begin{aligned} Y_t &= F(L_{t-1}) \\ F(0) &= 0 \\ F' &> 0 \\ F'' &< 0 \end{aligned} \tag{2.97}$$

Since wages are paid one period before the output is sold, the profit in period t is

$$\Pi_t = P_t Y_t - (1+i_t) W_{t-1} L_{t-1}$$

The first-order condition for profit-maximising employment is

$$P_t F'(L_{t-1}) = (1+i_t) W_{t-1} \tag{2.98}$$

Using the definitions:

$$1 + i_t \equiv (1+r_t)(1+\pi_t)$$

$$1 + \pi_t \equiv \frac{P_t}{P_{t-1}}$$

and

$$w_t \equiv \frac{W_t}{P_t}$$

equation 2.98 can be rewritten as

$$F'(L_{t-1}) = (1+r_t) w_{t-1} \tag{2.99}$$

One way to think about this is that the firm has to borrow to pay its workforce because the output and sales revenue produced by the workforce

only comes out of the pipeline with a lag. Since working capital is involved in the productive process, the real interest rate is part of the cost of labour. The old Austrians wrote extensively about capital as involving lags in production, roundaboutness etc. Models with lags between inputs and outputs are therefore sometimes referred to as 'Austrian' models. A useful reference is Blinder [6]

We revert to the simple labour supply function

$$\begin{aligned} w &= \ell(L) \\ \ell' &\geq 0 \end{aligned} \tag{2.100}$$

Dropping the time subscripts (that is, cheating a little), 12.58 and 2.100 imply that equilibrium employment is a decreasing function of the real interest rate:

$$\ell(L) = \frac{F'(L)}{1+r}$$

or

$$\begin{aligned} L &= j(r) \\ j' &< 0 \end{aligned}$$

We can write the IS-LM, AD-AS model with the Austrian supply side as follows:

$$c\left(F\left(j(i-\hat{\pi})-\bar{T}, i-\hat{\pi}, \frac{M+B}{P}\right)+\iota(i-\hat{\pi}, F\left(j(i-\hat{\pi})+G = F\left(j(i-\hat{\pi}\right)\right.\right.\right. \tag{2.101}$$

$$h\left[i, F\left(j(i-\hat{\pi})\right), \frac{M+B}{P}\right] = \frac{M}{P} \tag{2.102}$$

Exercise 2.2.2 *Check that the effect of an increase in public spending is (a) to raise the rate of interest, (b) to reduce employment and output, and (c) to raise the price level. Provide an intuitive explanation and show these effects graphically, in terms of shifts of the IS and LM curves and the AD and AS curves.*

2.2.2 Nominal Interest Rate Pegging

Consider again the case where the authorities peg i rather than M. Output is exogenous. For simplicity we omit the capital stock. The two endogenous variables are the price level, P, and the nominal money stock, M. Note again that, in the short-run, the total financial liabilities of the government, $L \equiv M + B$, is predetermined. This means that government open market operations and household portfolio reshuffles at a point in time are constrained by

$$dL = dM + dB = 0$$

The model's equilibrium conditions are given below.

$$c\left(Y(1-\tau) - \bar{T}, i - \hat{\pi}, \frac{L}{P}\right) + \iota(i - \hat{\pi}, Y) + G = Y$$

$$h(i, Y, \frac{L}{P}) = \frac{M}{P}$$

A point to note is that if the stock of nominal government bonds, B, equals zero, that is, if $L = M$, this model has appears to have *price level indeterminacy*. It appears that, with $B = 0$, the LM curve and the IS curve only depend on the real money stock $\frac{M}{P}$. This well-determined real money stock can be made up of infinitely many combinations of nominal money stocks and price levels. There is no nominal anchor in the flexible price model when the nominal interest rate is pegged. If $B \neq 0$, the total stock of nominally denominated public debt, $L \equiv M + B$ provides a nominal anchor and the equilibrium determines not only the real money stock, but also the nominal money stock and the price level separately.

While the argument seems plausible, it is wrong in the model under consideration. The reason is that with $L = M + B$ predetermined, B and M are both endogenous. We cannot impose $B = 0$ exogenously, because, with L predetermined, this would amount to setting M exogenously. M, of course, is endogenous. Even if the *equilibrium* value of B were zero, for some particular configuration of parameter values and exogenous variables, the model generates determinate values for the nominal money stock and the price level. There are two exceptional parameter configurations for which indeterminacy does prevail. The first is if there were no wealth effects in the consumption function and in the money demand function, $c_a = h_a = 0$. The second is

if a change in the general price level does not change real financial wealth, that is $L = 0$ or $M = -B$. The multipliers make this clear. It is convenient to choose the real stock of money balances and the price level as the two endogenous variables. The multipliers are found from:

$$\begin{bmatrix} -c_a\left(\frac{L}{P^2}\right) & 0 \\ \frac{h_a L}{P^2} & 1 \end{bmatrix} \begin{bmatrix} dP \\ d\left(\frac{M}{P}\right) \end{bmatrix} =$$

$$\begin{bmatrix} -(c_r+\iota_r) & -\frac{c_a}{P} & 1-(1-\tau)c_{Y_d}-\iota_Y & -1 & Yc_{Y_d} & c_{Y_d} & c_r+\iota_r \\ h_i & \frac{h_a}{P} & h_Y & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} di \\ dL \\ dY \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix} \tag{2.103}$$

where

$$\mu \equiv 1 - ((1-\tau)c_{Y_d} + \iota_Y) > 0$$

The determinant of the matrix $\begin{bmatrix} -c_a\left(\frac{L}{P^2}\right) & 0 \\ \frac{h_a L}{P^2} & 1 \end{bmatrix}$, denoted D is $D = -c_a\left(\frac{L}{P^2}\right) < 0$ unless $c_a = 0$ or $L = 0$, in which case there is not *nominal indeterminacy*, but an *overdeterminacy* of the real variables in the model. With $c_a = 0$ or $L = 0$, the model can be rewritten as

$$c\left(Y(1-\tau) - \bar{T}, i - \hat{\pi}, 0\right) + \iota(i - \hat{\pi}, Y) + G = Y \tag{2.104}$$

$$h(i, Y, 0) = \frac{M}{P}$$

Clearly, 2.104 cannot be satisfied in general, for exogenously given values of i, π, Y, τ, $\bar{T}$, and G.

Ruling out the conditions under which the determinant is zero, we get the following illustrative multipliers:

$$\frac{dP}{di} = \frac{-(c_r + \iota_r)}{D} < 0$$

$$\frac{d\left(\frac{M}{P}\right)}{di} = \frac{-c_a\left(\frac{L}{P^2}\right)h_i + h_a L(c_r + \iota_r)}{D} \lesseqgtr 0$$

A higher interest rate lowers the price level. If the wealth effect on the demand for real money balances is small, it will also reduce the demand for real money balances.

$$\frac{dP}{dG} = \frac{-1}{D} > 0$$

$$\frac{d\left(\frac{M}{P}\right)}{dG} = \frac{\frac{L}{P^2}h_a}{D} < 0$$

A higher level of public spending raises the price level and, through the wealth effect on money demand, reduces the real money stock.

Now consider what would happen if government bonds were index-linked rather than nominally denominated. Let b be the outstanding stock of one-period index-linked public debt. In this case $L \equiv M + Pb$. The model becomes

$$c\left(Y(1-\tau) - \bar{T}, i - \hat{\pi}, \frac{M}{P} + b\right) + \iota(i - \hat{\pi}, Y) + G = Y \tag{2.105}$$

$$h(i, Y, \frac{M}{P} + b) = \frac{M}{P} \tag{2.106}$$

Now it is no longer L that is predetermined, since it depends on the endogenous value of the price level. What constrains government open market operations and household portfolio reshuffles at a point in time is

$$dM + Pdb = 0 \tag{2.107}$$

The multipliers for the model given by 2.105, 2.106 *and* 2.107, can be found from 2.108.

$$\begin{bmatrix} -c_a\left(\frac{M}{P^2}\right) & 0 \\ \frac{(1-h_a)M}{P^2} & \frac{1}{P} \end{bmatrix} \begin{bmatrix} dP \\ dM \end{bmatrix} =$$

$$\begin{bmatrix} -(c_r+\iota_r) & 1-(1-\tau)c_{Y_d}-\iota_Y & -1 & Yc_{Y_d} & c_{Y_d} & c_r+\iota_r \\ h_i & h_Y & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} di \\ dY \\ dG \\ d\tau \\ d\bar{T} \\ d\hat{\pi} \end{bmatrix} \tag{2.108}$$

The determinant of the matrix $\begin{bmatrix} -c_a\left(\frac{M}{P^2}\right) & 0 \\ \frac{(1-h_a)M}{P^2} & \frac{1}{P} \end{bmatrix}$ is $-c_a\left(\frac{M}{P^3}\right) < 0$ unless $c_a = 0$ or $M = 0$. If either of these conditions holds, we have an overdetermined system. Nominal indeterminacy is never a problem, even if the debt is index-linked.

The conditions under which nominal indeterminacy prevails under a nominal interest rate peg amount to conditions that make the government's non-monetary debt (whether nominally denominated or index-linked) 'disappear' from the model, without this implying that nominal money stock is predetermined. This means that government debt is held by private agents, in an accounting sense, but does not influence private behaviour. These are, of course, the conditions under which debt neutrality or Ricardian equivalence holds. It should not be surprising that Ricardian equivalence is a necessary and sufficient condition for price level indeterminacy under a nominal interest rate peg. Without going through a full intertemporal optimising exercise, the mechanism can be made plausible. Under Ricardian equivalence, the wealth concept or permanent income concept that enters the private consumption function and asset demand functions is that of a representative consumer, whose planning horizon is as long as that of the fiscal-financial-monetary institutions in the economy.

Assume there is a representative infinite-lived consumer. We can write the consumption function as

$$\begin{aligned} C &= c(r, a+H) \\ c_{a+H} &> 0 \end{aligned}$$

As before, a is real financial wealth, that is,

$$a \equiv \frac{M+B}{P} + b$$

For sake of generality, we allow the consumer to hold both nominally denominated and index-linked public debt.

The second component of comprehensive private wealth is human capital, H, which is the present discounted value of all future after-tax labour income. Let y_L denote before-tax labour income and τ_L labour income taxes, which we assume to be the only taxes in the economy. The taxes are assumed to be lump-sum, that is, non-distortionary.

$$H(t) = \int_t^{\infty} e^{-\int_t^s r(u)ds}[y_L(s) - \tau_L(s)]ds \tag{2.109}$$

The government is subject to an intertemporal budget constraint or solvency constraint, which says that the value of its outstanding non-monetary liabilities equals the present discounted value of its future primary surpluses plus seigniorage revenues:

$$\frac{B(t)}{P(t)} + b(t) = \int_t^{\infty} e^{-\int_t^s r(u)ds}[\tau_L(s) - G(s) + \frac{\dot{M}(s)}{P(s)}]ds \tag{2.110}$$

The government is assumed to keep constant its real spending sequence and its sequence of real seigniorage revenue. It therefore adjusts the sequence of future lump-sum taxes so as to always satisfy its intertemporal budget constraint, which has to hold as an identity. The present discounted value of future taxes is therefore an endogenous variable, the residual that permits the government to satisfy its solvency constraint. The timing of the taxes does not matter in a representative agent model with non-distortionary taxes.

Substituting2.109 and 2.110 into the consumption function gives

$$C(t) = c\left(r(t), \frac{M(t)}{P(t)} + \int_t^{\infty} e^{-\int_t^s r(u)ds}[y_L(s) + \frac{\dot{M}(s)}{P(s)} - G(s)]ds\right) \tag{2.111}$$

We now wave our hands (this can be made rigorous) and assume that the government holds constant its future sequences of real public spending and real seigniorage (we also have to assume that the sequence of future real interest rates and of future before-tax labour income are independent of

what we are doing to the composition of the public debt, but this will indeed be the case). This means that $\int_t^{\infty} e^{-\int_t^s r(u)ds}[y_L(s) + \frac{\dot{M}(s)}{P(s)} - G(s)]ds = \Xi$. If something happens to change the real value of the outstanding stock of non-monetary liabilities, the sequence of future lump-sum taxes will adjust endogenously (because the government has to satisfy its intertemporal budget constraint), so as to leave total wealth While not necessarily $a+H$ unchanged. Government debt is private wealth, that is, it is a private asset, but it is cancelled out, behaviourally, by the present discounted value of future lump-sum taxes. . While not necessarily constant over time, Ξ is independent of the composition of the government's financial liabilities. The IS and LM model with the new consumption function and money demand function can now be written as

$$c(r, \frac{M}{P} + \Xi) + \iota(r, Y) + G = Y$$

and

$$\frac{M}{P} = h(i, Y, \frac{M}{P} + \Xi)$$

Thus $\frac{B}{P} + b$ have been behaviourally removed from the model. However, the government bonds are still there in an accounting sense, so open market operations are possible, with

$$dM + dB + PdB = 0$$

The initial money stock is therefore not predetermined. We now have a system that is well-posed, and allows us, under a nominal interest rate peg, to solve for the real equilibrium values of the endogenous variables $\frac{M}{P}$, but not for either the nominal money stock or the price level.

Thus price level indeterminacy under a nominal interest rate peg will prevail if and only if the model exhibits debt neutrality or Ricardian equivalence.

Chapter 3

The AD-AS Model in the Open Economy

3.1 The Fixed Price Case

Chapter 4

The Short-Run Aggregate Supply Curve

There are approximately 783 ways of deriving an aggregate supply curve that is upward-sloping in P, Y space. This chapter presents a small selection.

4.1 Disequilibrium Wage and Price Dynamics.

We return to the old labour demand and supply equations, again with the capital stock and distortionary taxes omitted. As in the simple classical model, both the demand for and the supply of labour depend on the real wage only. The difference with the previous versions is that the money wage is assumed to be sticky or rigid in a downward direction, at least in the short run. In each period there is an *inherited or predetermined* money wage, $\bar{W}$. As before, labour demand and supply are given by. The *actual* money wage is denoted W, as before.

$$\frac{W}{P} = F'(L^d)$$

$$\frac{W}{P} = \ell(L^s)$$

Let W^* be the *equilibrium*, market-clearing or full-employment money wage, that is the money wage that would clear the labour market, and L^* the market-clearing level of employment, defined by

$$F'(L^*) = \ell(L^*)$$

It follows that

$$W^* = PF'(L^*) = P\ell(L^*)$$

The equilibrium real wage, w^*, is given by

$$w^* = F'(L^*) = \ell(L^*)$$

The equilibrium or full-employment price level, P^*, is given by

$$P^* = \frac{\bar{W}}{w^*}$$

Equilibrium or full employment output, Y^*, is the level of output corresponding to the full employment level of employment, that is

$$Y^* = F(L^*)$$

Instead of assuming that the labour market always clears, we now assume that there is downward nominal rigidity in the wage: if the inherited or predetermined money wage is above the market clearing level, it will stay at that level, at least in the short run. With labour demand below labour supply, actual employment is assumed to be equal to labour demand. This is an application of the 'principle of voluntary exchange' or the 'short side rules' rule. If the inherited or predetermined money wage is below the equilibrium money wage, the money wage becomes flexible, and the actual money wage is the market-clearing or equilibrium money wage. This means there is downward money wage rigidity but not upward money wage rigidity.

Summarising, if the inherited money wage, $\bar{W}$, exceeds the equilibrium money wage, W^*, the actual money wage equals the inherited money wage and employment is given by labour demand (the smaller of labour demand and labour supply at the inherited money wage). If the inherited money wage is below the equilibrium money wage, the actual money wage equals the equilibrium money wage.

$$\begin{gathered} If\ \bar{W} > W^* \\ then \\ W = \bar{W} \\ and \\ L = L^d \end{gathered}$$

$$\begin{gathered} If\ \bar{W} \leq W^* \\ then \\ W = W^* \\ and \\ L = L^* \end{gathered}$$

This produces an Aggregate Supply Curve that is upward-sloping for $P < P^*$.(that is, when at the predetermined money wage, $L^d < L^s$). When $P \geq P^*$, the money wage becomes flexible, the labour market clears, and the aggregate supply curve becomes vertical at $Y = Y^*$.

When the AS curve is upward-sloping, it can be written as

$$\begin{aligned} P &= a(Y, \bar{W}) \\ a_Y &= \frac{-\bar{W}F''}{(F')^3} > 0 \\ a_{\bar{W}} &= \frac{1}{F'} > 0 \end{aligned}$$

The aggregate supply curve therefore shift up when the predetermined money wage, $\bar{W}$, increases.

A simple disequilibrium money wage adjustment equation makes the growth rate of money wages, $\frac{\Delta W}{W}$, an increasing function of the degree of excess demand in the labour market:

$$\begin{aligned} \frac{\Delta W}{W} &= \varphi(L^d - L^s) \\ \varphi' &> 0 \\ \varphi(0) &= 0 \end{aligned} \tag{4.1}$$

This is the original, non-expectations-augmented Phillips curve (actually the Fisher Curve). Phillips proxied excess demand in the labour market as a negative function of the unemployment rate, u. His original curve was

$$\begin{aligned} \frac{\Delta W}{W} &= g(u) \\ g' &< 0 \\ g'' &< 0 \end{aligned}$$

The rate of money wage inflation was decreasing in the unemployment rate, and the relationship was strictly convex: successive equal increments in

the unemployment rate were assumed to have weaker and weaker effects on the rate of growth of money wages.

With a money wage adjustment equation like 4.1, the money wage will be falling as long as the economy is on the upward-sloping part of the AS curve, since along the upward-sloping part of the AS curve, $L^d < L^s$.

4.2 The 'Surprise Supply Function'

An alternative approach to the short-run aggregate supply curve maintains the assumption that the labour market clears. The money wage is fully flexible. Labour demand and labour supply also depend on real wages only. The difference between the simple classical model and this one is that a distinction is made between the *real product wage*, which is relevant to labour demand and the *real consumption wage*, which is relevant to labour supply. As regards labour demand, the marginal product of labour is equated to the real product wage, the money wage deflated by the price of the firm's output). Firms are assumed to know exactly what the price of their output is.

$$F'(L^d) = \frac{W}{P}$$

Labour supply depends on the worker's perceived or expected real consumption wage (the money wage deflated by the price index of the consumption bundle consumed by the worker). This may contain hundreds of goods and services, and the worker does not necessarily know is current true value. We denoted the perceived or expected price level by $\hat{P}$.

$$l(L^s) = \frac{W}{\hat{P}}$$

$$L^d = L^s = L$$

The 'full information' real wage, w^*, employment, L^*, and output, Y^*, levels are the levels that prevail when $P = \hat{P}$. Thus

$$F'(L^*) = l(L^*)$$

$$w^* = F'(L^*)$$

$$Y^* = F(L^*)$$

It follows that actual employment in an increasing function of the ratio of the actual price level to the expected price level. When the actual price level increases, with the expected price level constant, the actual real wage falls, but the worker's expected real wage rises. Both labour demand and labour supply increase (note that, as a function of the actual real wage, labour supply can be written as

$$l(L^s)\frac{\hat{P}}{P} = \frac{W}{P}$$

From the labour market-clearing condition

$$F'(L) = \frac{\hat{P}}{P}l(L)$$

we get

$$\begin{gathered} L = L^* + \eta(\tfrac{P}{\hat{P}}) \\ \eta' > 0 \\ \eta(1) = 0 \end{gathered}$$

Output supply also is increasing in the ratio of the actual price to the expected price

$$\begin{gathered} Y = Y^* + \psi(\tfrac{P}{\hat{P}}) \\ \psi' > 0 \\ \psi(1) = 0 \end{gathered} \tag{4.2}$$

Holding constant the expected price level, the AS curve is upward-sloping in P, Y space. A higher expected price level shifts the AS curve to the right. It is possible to rewrite this 'surprise supply function' as an expectations-augmented price Phillips curve.

Let $p \equiv \ln P$. The rate of inflation, $\pi \equiv \frac{\Delta P}{P} \simeq \Delta p$.

A simple linear representation of 4.2 is

$$\begin{gathered} Y = Y^* + \tfrac{1}{\alpha}(p - \hat{p}) \\ \alpha > 0 \end{gathered}$$

or

$$p = \hat{p} + \alpha(Y - Y^*)$$

Rewrite this as

$$p - p_{-1} = \hat{p} - p_{-1} + \alpha(Y - Y^*)$$

or

$$\pi = \hat{\pi} + \alpha(Y - Y^*) \tag{4.3}$$

where $\hat{\pi}$ is the expected rate of inflation.

In a diagram with π on the vertical axis and Y on the horizontal axis, we can draw an upward-sloping short-run price Phillips curve for every expected rate of inflation. A higher expected rate of inflation shifts this short-run price Phillips curve up vertically by the same magnitude as the increase in the expected rate of inflation.

If expected inflation is given by the simple mechanical rule

$$\hat{\pi} = \pi_{-1}$$

we get the simple accelerationist price Phillips curve:

$$\Delta\pi = \alpha(Y - Y^*) \tag{4.4}$$

4.3 The 'Thousand Islands' Theory

The following theory of the short-run aggregate supply curve is due to Lucas [21] [**?**]. There are lots (N) of uncoordinated little producers, working and producing on isolated islands. There is no telephone. In period t, each producer, i, bases his production decision on the 'local' price in his island, p_t^i, relative to the average price in all the islands, p_t (there really should be an intertemporal price here, but never mind). The local price is private knowledge. Producers are randomly reallocated over the islands each period. Each producer has to make an informed guess about the general or average price level. The rational expectation of p in period t is denoted $E_t p_t$.

$$y_t^i = y_t^{*i} + \beta(p_t^i - E_t p_t)$$
$$\beta > 0$$

$$p_t \equiv \frac{\sum_{j=1}^{N} p_t^i}{N}$$

To make an estimate of p_t in period t, there are 2 bits of information, the local price, p_t^i, and the best guess of p_t based on prior information (based period $t-1$ or earlier).

Therefore, using a linear predictor, this signal extraction problem has the following solution:

$$E_t p_t = \gamma p_t^i + (1-\gamma)E_{t-1}p_t$$

where the weight given to local price information is given by

$$\gamma = \frac{\sigma^2}{\sigma^2 + \sigma_i^2}$$

where σ^2 is the variance of the general price level (roughly the variance of aggregate monetary or nominal demand shocks) and σ_i^2 is the variance of the local price (roughly the variance of the local demand shock).

For producer i we therefore obtain the following supply curve

$$y_t^i = y_t^{*i} + \beta(1-\gamma)(p_t^i - E_{t-1}p_t)$$
$$\beta > 0$$

Average aggregate supply, $y \equiv \frac{\sum_{j=1}^{N} y_t^i}{N}$, is given by:

$$y_t = y_t^* + \beta\left(\frac{\sigma_i^2}{\sigma^2 + \sigma_i^2}\right)(p_t - E_{t-1}p_t)$$

We can rewrite this as a surprise supply function

$$p_t = E_{t-1}p_t + \beta^{-1}\left(\frac{\sigma^2 + \sigma_i^2}{\sigma_i^2}\right)(y_t - y_t^*)$$

When the variance of aggregate demand is large relative to the variance of the local shocks (that is, when aggregate nominal shocks are a poor signal for local shocks), the AS curve will be steep in P, Y space. When .the variance of aggregate demand is small relative to the variance of the local shocks (that is, when aggregate nominal shocks are a good signal for local shocks), the AS curve will be flat in P, Y space. When an observed price level increase is most likely to be part of a general inflationary increase in the price level, a weak supply response is optimal. When an observed price level increase is most likely to be a purely local relative price shock, a strong supply response is optimal.

4.4 The Phillips Curve in a Closed Economy

Let $w \equiv \ln W$. The rate of money wage inflation is denoted $\pi_w \equiv \frac{\Delta W}{W} \simeq \Delta w$.

The simplest augments wage Phillips curve is given by

$$\begin{gathered}\pi_w(t) = \gamma(t) - \alpha u(t) + \bar{\pi}(t) \\ \alpha > 0\end{gathered} \tag{4.5}$$

The interpretation of γ is the *target* growth rate of real wages. The augmentation term in the wage Phillips curve, $\bar{\pi}$, is *core inflation.* It can be interpreted as expected inflation, but needs not be restricted to that. Other influences, such as long-term nominal contracting, can impart momentum to the wage inflation process.

We assume that prices are a constant proportional mark-up on unit variable cost. In the closed economy, unit variable cost is unit labour cost, therefore

$$\begin{aligned}P &= \mu\frac{WL}{Y} \\ \mu &> 0\end{aligned} \tag{4.6}$$

With μ constant, it follows that

$$\pi = \pi_w - \gamma_L \tag{4.7}$$

where γ_L is the actual growth rate of average labour productivity

$$\bar{\gamma} = \Delta \ln Y - \Delta \ln L$$

An alternative price equation can be derived from the competitive first-order condition

$$P = \frac{W}{MPL} \tag{4.8}$$

where MPL is the marginal product of labour. This implies

$$\pi = \pi_w - \gamma_{MPL}$$

where γ_{MPL} is the growth rate of the marginal product of labour.
From 4.7 and 4.5 it follows that:

$$\pi = \bar{\pi} + \gamma - \gamma_L - \alpha u \tag{4.9}$$

The natural rate of unemployment or NAIRU (for non-accelerating rate of inflation rate of unemployment), u_N, is the rate of inflation that prevails when actual inflation equals core inflation ($\pi = \bar{\pi}$), that is,

$$u_N = \alpha^{-1}(\gamma - \gamma_L) \tag{4.10}$$

The natural rate will be lower the higher α, that is the more responsive wage inflation is to excess demand in the labour market, and the smaller $\gamma - \gamma_L$, the gap between the target growth rate of real wages and the 'warranted' growth rate of real wages.

Using 4.10 we can rewrite 4.9 as

$$\pi = \bar{\pi} - \alpha(u - u_N) \tag{4.11}$$

An alternative, and empirically more plausible price mark-up equation makes P a constant proportional mark up not on actual unit variable cost but on 'normal unit variable cost', $\overline{WL/Y}$. Normal unit variable cost can and has been defined in many different ways. All definitions try to get at some notion of long-run average unit variable cost, either backward-looking or forward-looking (expected future long-run unit variable cost). The price equation becomes

$$\begin{gathered} P = \mu \overline{WL/Y} \\ \mu > 0 \end{gathered} \tag{4.12}$$

and

$$\pi = \bar{\pi}_w - \bar{\gamma}_L \tag{4.13}$$

The closed economy price Phillips curve can therefore be written as

$$\pi = \bar{\pi} + \gamma - \gamma_L - \alpha u + [\gamma_L - \bar{\gamma}_L + \bar{\pi}_w - \pi_w] \tag{4.14}$$

Compared with 4.9, equation 4.14 implies that price inflation will be lower when current labour productivity growth is below-trend ($\gamma_L < \bar{\gamma}_L$), which is likely to be the case during the downturn of the business cycle, and when current earnings growth is above normal ($\pi_w > \bar{\pi}_w$).

It now becomes useful to distinguish between the short-run natural rate of unemployment or short-run NAIRU, u_N, and the long-run natural rate of unemployment or long-run NAIRU, $\bar{u}_N$.

The short-run natural rate is the unemployment rate that prevails when actual inflation equals core inflation, $\pi = \bar{\pi}$. This implies

$$u_N = \alpha^{-1}\left(\gamma - \gamma_L + [\gamma_L - \bar{\gamma}_L + \bar{\pi}_w - \pi_w]\right) \tag{4.15}$$

Note that, because of the likely cyclical behaviour of $\gamma_L - \bar{\gamma}_L$ and $\bar{\pi}_w - \pi_w$, the short-run natural rate is likely to be a cyclical variable.

The long-run natural rate of unemployment is the unemployment rate that prevails when actual inflation equals core inflation *and* normal unit variable cost inflation equals actual unit variable cost inflation: $\pi = \bar{\pi}$ and $\gamma_L = \bar{\gamma}_L$ and $\pi_w = \bar{\pi}_w$.

$$\bar{u}_N = \alpha^{-1}(\gamma - \bar{\gamma}_L) \tag{4.16}$$

We can therefore still write the closed economy price Phillips curve as in 4.11. It is probably more informative (from the point of view of separating out long-run structural factors and short-run cyclical factors) to write it in terms of deviations from the long-run natural rate as follows

$$\pi = \bar{\pi} - \alpha(u - \bar{u}_N) + [\gamma_L - \bar{\gamma}_L + \bar{\pi}_w - \pi_w] \tag{4.17}$$

4.5 The Phillips Curve in an Open Economy

The wage Phillips curve has the same form as for the closed economy. Note, however, that the core inflation term in the open economy wage Phillips curve, is likely to involve the (core) inflation rate of a cost -of-living index, that is something like the consumer price index (CPI) or the retail price index

(RPI) rather than the (core) inflation rate of the GDP deflator or the (gross) price of domestically produced goods and services. The rate of inflation of the consumer price index is denoted $\overline{\tilde{\pi}}(t) \equiv \frac{\Delta \tilde{P}}{\tilde{P}}$, where $\tilde{P}$ is the cost-of living index, the rate of inflation of the gross price of domestic output (not the GDP deflator) is denoted $\pi \equiv \frac{\Delta P}{P}$, the rate of inflation of world prices (in foreign currency) is denoted $\pi^* \equiv \frac{\Delta P^*}{P^*}$, where P^* is the foreign price level, and the depreciation rate of the nominal exchange rate is denoted $\varepsilon \equiv \frac{\Delta E}{E}$ where E is the nominal spot exchange rate (the number of units of home currency per unit of foreign currency)..

$$\begin{gathered} \pi_w(t) = \gamma(t) - \alpha u(t) + \overline{\tilde{\pi}}(t) \\ \alpha > 0 \end{gathered} \tag{4.18}$$

$$\begin{gathered} \tilde{\pi}(t) = \varpi\pi + (1-\varpi)(\pi^* + \varepsilon) \\ 1 \geq \varpi \geq 0 \end{gathered} \tag{4.19}$$

The share of imports in the consumption bundle is denoted ϖ.

The (gross) price of domestic output is assumed to be a constant proportional mark-up on unit variable cost, the sum of unit labour cost and unit import cost. The volume of imported intermediate and raw materials imports is denoted N.

$$P = \mu\left(\frac{WL + EP^*N}{Q}\right) \tag{4.20}$$

Note the distinction between Y, which denotes home country GDP, and Q , which is home country gross output (including the value of imported intermediate and raw materials imports). Equation 4.20 implies that

$$\begin{gathered} \pi = \lambda(\pi_w - \gamma_L) + (1-\lambda)(\pi^* + \varepsilon - \gamma_N) \\ 1 \geq \lambda \geq 0 \end{gathered} \tag{4.21}$$

The share of labour cost in total variable cost is denoted λ, .and the growth rate of average labour productivity now is given by $\gamma_L \equiv \frac{\Delta(Q/L)}{Q/L}$, while the growth rate of average import productivity is given by $\gamma_N \equiv \frac{\Delta(Q/N)}{Q/N}$.

This implies the following Phillips curve for the cost-of-living index:

$$\tilde{\pi} = \overline{\tilde{\pi}} + \varpi\lambda(\gamma - \gamma_L - \alpha u) + (1-\varpi\lambda)(\pi^* + \varepsilon - \overline{\tilde{\pi}}) - \varpi(1-\lambda)\gamma_N \tag{4.22}$$

The natural rate of unemployment is again the unemployment rate that prevails when core inflation equals actual inflation, $\tilde{\pi} = \bar{\bar{\pi}}$. This implies

$$u_N = \frac{1}{\alpha}\left[\gamma - \gamma_L - \left(\frac{1-\lambda}{\lambda}\right)\gamma_N + \left(\frac{1-\varpi\lambda}{\varpi\lambda}\right)(\pi^* + \varepsilon - \tilde{\pi})\right] \tag{4.23}$$

The natural rate will again be lower the greater the value of α,,the responsiveness of wage inflation to unemployment, the lower the target growth rate of real wages, γ, and the higher the growth rates of labour productivity, γ_L and import productivity, γ_N. The natural rate will also be lower if the country's terms of trade are improving (that is, if its competitiveness is worsening). When domestic inflation exceeds world inflation $\tilde{\pi} > \pi^* + \varepsilon$, the worker's real consumption wage (which is defined in terms of the cost-of-living index $\tilde{P}$, rises relative to the real product wage that determines the demand for labour by firms, which is defined in terms of the domestic producer price index P. Workers' real wage aspirations can be satisfied to a greater extent through relatively cheaper import prices rather than through higher money wages.

Using this definition of the natural rate of unemployment, we can write the open economy price Phillips curve as

$$\tilde{\pi} = \bar{\bar{\pi}} - \varpi\lambda\alpha(u - u_N) \tag{4.24}$$

As in the case of the closed economy Phillips curve, an alternative, and empirically more plausible price mark-up equation makes P a constant proportional mark up not on actual unit variable cost but on 'normal unit variable cost', $\overline{(WL + EP^*)/Y}$. The normal cost price equation becomes

$$P = \mu\overline{(WL + EP^*)/Y} \qquad \mu > 0 \tag{4.25}$$

and

$$\pi = \lambda(\bar{\pi}_w - \bar{\gamma}_L) + (1-\lambda)(\bar{\pi}^* + \bar{\varepsilon} - \bar{\gamma}_N) \qquad 1 \geq \lambda \geq 0 \tag{4.26}$$

This implies the following open economy Phillips curve for the cost-of-living index:

$$\begin{aligned}\tilde{\pi} &= \bar{\bar{\pi}} + \varpi\lambda(\gamma - \gamma_L - \alpha u) + (1-\varpi\lambda)(\pi^* + \varepsilon - \bar{\bar{\pi}}) - \varpi(1-\lambda)\gamma_N \\ &+\varpi\lambda\left[\bar{\pi}_w - \bar{\gamma}_L - (\pi_w - \gamma_L)\right] + \varpi(1-\lambda)\left[\bar{\pi}^* + \bar{\varepsilon} - \bar{\gamma}_N - (\pi^* + \varepsilon - \gamma_N)\right]\end{aligned} \tag{4.27}$$

The normal unit variable cost pricing equation, 4.27, differs from the actual unit variable cost pricing equation 4.22, by having a higher rate of inflation when normal unit labour cost inflation exceeds current unit labour cost inflation, $\bar{\pi}_w - \bar{\gamma}_L > \pi_w - \gamma_L$ and/or when normal unit import cost inflation exceeds current unit import cost inflation, $\bar{\pi}^* + \bar{\varepsilon} - \bar{\gamma}_N > \pi^* + \varepsilon - \gamma_N$.

The short-run natural rate of unemployment is again the unemployment rate that prevails when core inflation equals actual inflation, $\tilde{\pi} = \bar{\bar{\pi}}$. This implies

$$\begin{aligned} u_N &= \tfrac{1}{\alpha}\left[\gamma - \gamma_L - \left(\tfrac{1-\lambda}{\lambda}\right)\gamma_N + \left(\tfrac{1-\varpi\lambda}{\varpi\lambda}\right)(\pi^* + \varepsilon - \tilde{\pi})\right] \\ &+ \tfrac{1}{\alpha}\left[\bar{\pi}_w - \bar{\gamma}_L - (\pi_w - \gamma_L) + \tfrac{(1-\lambda)}{\lambda}\left[\bar{\pi}^* + \bar{\varepsilon} - \bar{\gamma}_N - (\pi^* + \varepsilon - \gamma_N)\right]\right] \end{aligned} \tag{4.28}$$

Note again that the short-run natural rate is likely to be cyclical.

The long-run natural rate of unemployment $\bar{u}_N$ is the unemployment rate that prevails when actual inflation equals core inflation *and* normal unit variable cost inflation equals actual unit variable cost inflation: $\tilde{\pi} = \bar{\bar{\pi}}$ *and* $\gamma_L = \bar{\gamma}_L$, $\bar{\pi}_w = \pi_w$, $\bar{\pi}^* = \pi^*$, $\bar{\varepsilon} = \varepsilon$ and $\bar{\gamma}_N = \gamma_N$. It follows that

$$\bar{u}_N = \frac{1}{\alpha}\left[\gamma - \gamma_L - \left(\frac{1-\lambda}{\lambda}\right)\gamma_N + \left(\frac{1-\varpi\lambda}{\varpi\lambda}\right)(\pi^* + \varepsilon - \tilde{\pi})\right] \tag{4.29}$$

Using the definition of the short-run natural rate in4.28 we can still write the Phillips curve as in 4.24. It is again more informative, from the point of view is separating out long-run structural factors and short-run cyclical factors, to write the Phillips curve in terms of deviations from the long-run natural rate, as follows:

$$\begin{aligned} \pi &= \tilde{\pi} - \varpi\lambda\alpha(u - \bar{u}_N) \\ &+ \left[\varpi\lambda\left(\bar{\pi}_w - \bar{\gamma}_L - (\pi_w - \gamma_L)\right) + \varpi(1-\lambda)\left[\bar{\pi}^* + \bar{\varepsilon} - \bar{\gamma}_N - (\pi^* + \varepsilon - \gamma_N)\right]\right] \end{aligned} \tag{4.30}$$

Presentationally even more informative would be a definition of the natural rate that also controls for variations in the real exchange rate. The very long-run natural rate, $\bar{\bar{u}}_N$, is the unemployment rate that prevails when actual inflation equals core inflation *and* normal unit variable cost inflation equals actual unit variable cost inflation: $\tilde{\pi} = \bar{\bar{\pi}}$ *and* $\gamma_L = \bar{\gamma}_L$, $\bar{\pi}_w = \pi_w$, $\bar{\pi}^* = \pi^*$, $\bar{\varepsilon} = \varepsilon$ and $\bar{\gamma}_N = \gamma_N$, *and* the real exchange rate is constant $\pi^* + \varepsilon = \pi$., It follows that

$$\bar{\bar{u}} \equiv \frac{1}{\alpha}\left[\gamma - \gamma_L - \left(\frac{1-\lambda}{\lambda}\right)\gamma_N\right] \tag{4.31}$$

The open economy price Phillips curve can now be written as

$$\begin{aligned}
&\tilde{\pi} = \bar{\bar{\pi}} - \varpi\lambda\alpha(u - \bar{\bar{u}}_N) \\
&+ \left[\varpi\lambda\left(\bar{\pi}_w - \bar{\gamma}_L - (\pi_w - \gamma_L)\right) + \varpi(1-\lambda)\left[\bar{\pi}^* + \bar{\varepsilon} - \bar{\gamma}_N - (\pi^* + \varepsilon - \gamma_N)\right]\right] \\
&+ (1 - \varpi\lambda)(\pi^* + \varepsilon - \bar{\bar{\pi}})
\end{aligned} \tag{4.32}$$

4.6 The Cost of Disinflation: the Sacrifice Ratio

One useful measure of the output or employment cost of reducing inflation is the *sacrifice ratio.* This is defined as the *cumulative* increase in the unemployment rate (or reduction in output) required to achieve a one percentage point *sustained* reduction in the rate of inflation.

For the Phillips curve models, the sacrifice ratio depends on three key features. First, the responsiveness of wage inflation to unemployment. Second, the determinants of the core inflation rate, and in particular the degree of inertia or persistence in core inflation. Third, the determinants of the natural rate of unemployment, and especially the question as to whether the disinflation process itself can affect the natural rate. In what follows, I focus on a very simple representation of the Phillips curve.

$$\pi = \bar{\pi} - \alpha(u - u_N) \tag{4.33}$$

The responsiveness of (wage) inflation to unemployment is measured by the parameter α, $\alpha > 0$, and is treated as constant throughout.

4.6.1 An Exogenous Natural Rate

In this subsection, the natural rate is treated as exogenous and, for simplicity, constant. Note that a permanent reduction in the actual rate of inflation has been achieved when the core rate of inflation equals the actual rate of inflation at the new lower level. When that happens, the actual rate of unemployment can be set equal to the natural rate again and the lower actual and core rates of inflation will persist forever.

Exogenous Core Inflation

$$\bar{\pi} = \bar{\bar{\pi}} \tag{4.34}$$

This includes the original non-augmented Phillips curve as a special case with $\bar{\pi} = 0$. When core inflation does not respond at all to past, present or anticipated future actual inflation, a permanent increase in unemployment is required to ensure a permanent reduction in inflation. To get inflation lower by 1% for one period, unemployment that period will have to rise by $\frac{1}{\alpha}$. To keep inflation at the new lower level, unemployment will have to stay at that higher level. Assume the reference path for unemployment is $u = u_N$. The initial period is $t = 1$. The sacrifice ratio, σ , is

$$\sigma \equiv \sum_{t=1}^{\infty}(u(t) - u_N) = \sum_{t=1}^{\infty}\frac{1}{\alpha} = \infty \tag{4.35}$$

With exogenous core inflation and an exogenous natural rate, the sacrifice ratio is infinite.

Static Core Inflation

$$\bar{\pi}(t) = \pi(t-1) \tag{4.36}$$

In this case

$$u(t) - u_N = -\frac{1}{\alpha}\left(\pi(t) - \pi(t-1)\right) \tag{4.37}$$

Now we only need a temporary increase in unemployment in order to achieve a permanent reduction in inflation. In fact, by raising employment by $\frac{1}{\alpha}$ for one period only (say during period 1, we can reduce inflation that period by 1%. The next period (period 2) we can reduce unemployment back to its initial level (u_N). Core inflation in period 2 is the actual level of inflation during period 1. Inflation from period 2 on can therefore be kept permanently at the new lower level even though unemployment from period 2 on is back at the natural rate.

$$\sigma = \frac{1}{\alpha} \tag{4.38}$$

One possible interpretation of core inflation is *expected* inflation. The sacrifice ratio will be lower the greater the responsiveness of inflation to the unemployment rate, α.

Adaptive Core Inflation.

$$\bar{\pi}(t) = \eta\pi(t-1) + (1-\eta)\bar{\pi}(t-1), 0 \leq \eta \leq 1 \tag{4.39}$$

Using 4.33 to eliminate the actual inflation rate from 4.39, we get

$$\bar{\pi}(t) - \bar{\pi}(t-1) = -\alpha\eta(u(t-1) - u_N)$$

Remember that a permanent 1% change in the actual rate of inflation is achieved once there has been a change of 1% in the core rate of inflation. A1% change in the core rate of inflation requires a one-period increase in unemployment of $\frac{1}{\alpha\eta}$. Therefore

$$\sigma = \frac{1}{\alpha\eta} \tag{4.40}$$

The sacrifice ratio is lower the greater α, the responsiveness of inflation to unemployment and the greater η, the speed of adaptation of core inflation. Note that the two previous models of core inflation are special cases of the adaptive case.

Generalised Adaptive Core Inflation

$$\bar{\pi}(t) = \Lambda(L)\pi(t-1) = \sum_{i=0}^{\infty} \lambda_i \pi(t-1-i) \tag{4.41}$$

We want the augmented Phillips curve to have the *natural rate property*, that is,

$$\Lambda(1) \equiv \sum_{i=0}^{\infty} \lambda_i = 1$$

This means that if the actual rate of inflation is constant forever, the core rate of inflation will equal the actual rate of inflation. The long-run Phillips curve is vertical.

From 4.33 and 4.41 it follows that

$$\pi(t) = \Lambda(L)\pi(t-1) - \alpha(u(t) - u_N) \tag{4.42}$$

and that

$$\bar{\pi}(t+1) = \Lambda(L)\bar{\pi}(t) - \Lambda(L)\alpha(u(t) - u_N) \tag{4.43}$$

We now calculate the properties of sequences of unemployment that can bring the inflation rate from a given constant level π^* for $t < 0$ to another constant level π^{**} for $t \geq 0$. Tedious calculation implies that

$$\sigma \equiv \frac{\sum_{i=0}^{\infty}(u(i) - u_N)}{\pi^{**} - \pi^*} = \frac{1}{\alpha}\sum_{j=0}^{\infty}(1+j)\lambda_j \tag{4.44}$$

$\sum_{j=0}^{\infty}(1+j)\lambda_j$ is the mean lag of the lag operator $1 - L\Lambda(L)$.

The three previous core inflation processes are special cases of 4.41. Exogenous core inflation is the special case where $\lambda_i = 0$ for all i. Static core inflation is the special case where $\lambda_0 = 1$ and $\lambda_i = 0$ for $i > 0$. The adaptive core inflation process is the special case where $\lambda_i = \eta(1-\eta)^i$. As expected, the sacrifice ratio is lower the lower α and the shorter the mean lag in the core inflation process.

Mixed Backward-Looking and Forward-Looking Core Inflation

In the spirit, if not the letter of Taylor-style staggered, overlapping contract models, we could make the current core rate of inflation a function partly of past inflation and partly of anticipated future inflation. E_t is the expectation operator conditional on information at time t.

$$\begin{gathered}\bar{\pi}(t) = (1-\delta)E_t\pi(t+1) + \delta\pi(t-1) \\ 0 \leq \delta \leq 1\end{gathered} \tag{4.45}$$

Substituting this into 4.33 yields

$$\pi(t) = (1-\delta)E_t\pi(t+1) + \delta\pi(t-1) - \alpha(u(t) - u_N) \tag{4.46}$$

From 4.46 it is clear that, since $\pi(t+1)$ will depend on $u(t+1)$ and on period $t+1$ expectations of $\pi(t+2)$, the following guess solution seems reasonable

$$\pi(t) = A_0\pi(t-1) + B_0\left(u(t) - u_N\right) + \sum_{i=1}^{\infty} B_i E_t\left(u(t+i) - u_N\right) \tag{4.47}$$

The coefficients A_0 and $B_i,\ i \geq 0$ are found by substituting the guess solution into the model and equating coefficients between the resulting equation and the guess solution.[1]

The undetermined coefficients are determined by

$$A_0 = (1-(1-\delta)A_0)^{-1}\,\delta \tag{4.48}$$

$$B_0 = -\,(1-(1-\delta)A_0)^{-1}\,\alpha \tag{4.49}$$

$$B_j = (1-\delta)^j B_0\ ,\ j > 0 \tag{4.50}$$

We solve for the equilibrium value of A_0 from

$$A_0^2 - \frac{1}{1-\delta}A_0 + \frac{\delta}{1-\delta} = 0 \tag{4.51a}$$

This implies

$$\begin{array}{l} A_0 = \frac{\delta}{1-\delta} \\ A_0 = 1 \end{array} \tag{4.52}$$

When $\delta = 1,$ it is clear from equation 4.48 (and from economic common sense, see below), that only the solution $A_0 = 1$ is appropriate.

When there are two distinct roots, one of which is weakly stable (modulus less than or equal to 1) and one of which is unstable (modulus greater than 1), we would normally choose the stable one. Except when $\delta = 0,$ the rate of inflation, π, is a predetermined state variable (its initial value is inherited from the past and cannot respond to news about future events). We would normally like the homogeneous part of equation 4.47, that is, $\pi(t) = A_0\pi(t-1)$, to be non-explosive.

Note that, when $\delta = \frac{1}{2}$, that is, when future expected inflation has the same weight as past inflation, $A_0 = 1$. The homogeneous part of equation 4.47 has a unit root. $B_0 = -2\alpha$ and $B_j = -\left(\frac{1}{2}\right)^j 2\alpha,\ j \geq 1$.

[1] Use is made of the 'Law of Iterated Projections', $E_t E_{t+j}(x) = E_t(x),\ j \geq 0$, (the earlier expectation of a later expectation is the earlier expectation). This is an immediate application of a fundamental property of conditional expectations, provided that the information set conditioning expectations at an earlier date is not richer than the information set at a later date.

When $\delta = 0$, that is, when only expected future inflation affects current inflation and past inflation has no weight at all, equation 4.52 has two roots, $A_0 = 0$ and $A_0 = 1$. Since it makes no economic sense to have past inflation entering the solution of a purely forward-looking structural inflation process, we choose the solution $A_0 = 0$. In this case, $B_j = -\alpha \;, i \geq 0$.

When $\delta = 1$, that is, when only past inflation affects current inflation and expected future inflation has no weight at all, equation 4.52 has only one economically meaningful root, $A_0 = 1$. In this case, $B_0 = -\alpha$, and $B_j = 0 \; j > 0$.

For the first solution of 4.51a, $A_0 = \frac{\delta}{1-\delta}$, A_0 increases monotonically with δ over the interval $0 \leq \delta \leq 1$. However, for $\delta > \frac{1}{2}$, $A_0 > 1$. Since as $\delta \to 1$, $\frac{\delta}{1-\delta} \to \infty$, the second solution, $A_0 = 1$ is clearly the only reasonable one when $\delta = 1$. The problem is, for what value of δ do we switch from the first solution, $A_0 = \frac{\delta}{1-\delta}$, which is clearly the only sensible one when $\delta = 0$, to the second solution, $A_0 = 1$? On economic grounds I would argue that we should use $A_0 = \frac{\delta}{1-\delta}$ for $\delta < \frac{1}{2}$ and $A_0 = 1$ for $\delta \geq \frac{1}{2}$. I adopt this choice in what follows.

It is clear that, for all values of $\delta, 0 \leq \delta \leq 1$, the sacrifice ratio is finite: it only takes a temporary increase in unemployment to achieve a permanent reduction in inflation. It is also clear that the sacrifice ratio depends only on α and δ.

When $\delta \geq \frac{1}{2}$ (which implies $A_0 = 1$), it is always possible, since there only is one lag in the inflation process, and the coefficient on lagged inflation in 4.47 is unity, to achieve a permanently lower rate of inflation by increasing unemployment in just one period. Consider the case where unemployment in raised only in period t, that is, $u(t+i) = u_N \;, \; i \geq 1$. From 4.47 it then follows that

$$\sigma = -B_0^{-1} = \frac{\delta}{\alpha} \tag{4.53}$$

As expected, the sacrifice ratio is smaller when the responsiveness of inflation to unemployment is higher and when the relative weight of past inflation to expected future inflation is lower.

Forward-looking expectations and the case for gradualism For $\delta \geq \frac{1}{2}$,

$$\pi(t) = \pi(t-1) - \frac{\alpha}{\delta}\left(u(t) - u_N\right) - \frac{\alpha}{\delta}\sum_{i=1}^{\infty}(1-\delta)^i E_t\left(u(t+i) - u_N\right) \tag{4.54}$$

Consider the case where unemployment is increased by the same amount in periods t and $t+1$. For all subsequent periods, unemployment is kept at the natural rate. This means

$$\pi(t) = \pi(t-1) - \frac{\alpha}{\delta}\left(u(t) - u_N\right) - \frac{\alpha}{\delta}(1-\delta)\left(u(t+1) - u_N\right)$$

$$\begin{aligned}\pi(t+1) &= \pi(t) - \frac{\alpha}{\delta}\left(u(t+1) - u_N\right) \\ &= \pi(t-1) - \frac{\alpha}{\delta}\left(u(t) - u_N\right) - \frac{\alpha}{\delta}(2-\delta)\left(u(t+1) - u_N\right)\end{aligned}$$

Since unemployment is, by assumption, the same in periods t and $t+1$,

$$u(t) - u_N = u(t+1) - u_N = \bar{u} - u_N$$

Therefore,

$$\pi(t+1) = \pi(t-1) - \frac{\alpha}{\delta}(3-\delta)\left(\bar{u} - u_N\right)$$

The unemployment gap we need to maintain for 2 periods in order to reduce inflation by 1% in two periods (and keep it at that lower level forever after) is therefore given by

$$u - u_N = \frac{\delta}{\alpha(3-\delta)}$$

The sacrifice ratio is

$$\sigma = \frac{2\delta}{\alpha(3-\delta)} < \frac{\delta}{\alpha} \text{ for } \delta < 1$$

This is plausible. Unemployment in period $t+1$ works twice. First in period $t+1$ directly, and second during period t, when the expectation of period $t+1$ unemployment depresses period t inflation. Since policy has both direct effects and announcement effects when expectations are forward-looking, there is a case for spreading out the policy over time rather than applying one dose immediately, that is, a case for gradualism rather than cold turkey. Note that there may be a credibility problem here. Once the

(credible) announcement (in period t) of higher unemployment in period $t = 1$ has had its desired effect on inflation in period t, the authorities may no longer wish to incur the higher unemployment when period $t+1$ arrives. Time inconsistency problems of this kind arise whenever policy has announcement effects.

If the authorities in period t wish to reduce inflation permanently starting in period $t+1$, they could also do so by just increasing unemployment in period $t+1$.

$$\pi(t) = \pi(t-1) - \frac{\alpha}{\delta}(1-\delta)\left(u(t+1) - u_N\right)$$

$$\begin{aligned}\pi(t+1) &= \pi(t) - \frac{\alpha}{\delta}\left(u(t+1) - u_N\right) \\ &= \pi(t-1) - \frac{\alpha}{\delta}(2-\delta)\left(u(t+1) - u_N\right)\end{aligned}$$

In this case the sacrifice ratio is

$$\sigma = \frac{\delta}{\alpha(2-\delta)} < \frac{2\delta}{\alpha(3-\delta)} < \frac{\delta}{\alpha} \text{ for } \delta < 1$$

As expected, concentrating the necessary unemployment in period $t+1$ results in an even lower sacrifice ratio than spreading it evenly over periods t and $t+1$.

There are two qualifications. First, the sacrifice ratio sums the undiscounted increases in unemployment.. The authorities may not in fact be indifferent about the timing of unemployment. Second, the inflation profiles are different for the three cases, before period $t+1$. Inflation in period t is lowest when the unemployment is all concentrated in period t and highest when it is all concentrated in period $t+1$.

Exercise 4.6.1 *Check, for the case where $\delta \geq \frac{1}{2}$, whether the sacrifice ratio depends on the timing of the unemployment increases. Derive, specifically, the sacrifice ratio when unemployment is increased in only one* future *period, and the sacrifice ratio when unemployment is increased in the current period and the next. Explain your answers.*

Costless Disinflation: the Surprise Supply Function and Rational Expectations

Our final example equates core inflation in period t with the rate of inflation expected, in period $t-1$, to prevail in period t, that is

$$\bar{\pi}(t) = E_{t-1}\pi(t) \tag{4.55}$$

Rational, or model-consistent expectations imply that

$$\pi(t) = E_{t-1}\pi(t) + \varepsilon(t-1,t)$$

where $\varepsilon(t-1,t)$ is the rational forecast error in period t for forecasts made in period $t-1$. Since $\varepsilon(t-1,t)$ is a rational forecast error,

$$E_{t-1}\varepsilon(t-1,t) = 0$$

Inflation is no longer a predetermined state variable. It can adjust instantaneously and costlessly to credible announcements about future policy. It follows that the sacrifice ratio is zero: costless disinflation only requires credibility

$$\sigma = 0$$

This is the counterpart of the proposition that, with a surprise supply function and rational expectations, policy cannot influence the first moment of the distribution of real output, employment or unemployment:

$$u(t) = u_N + \alpha^{-1}\varepsilon(t-1,t)$$

4.6.2 Hysteresis in the Natural Rate of Unemployment

Hysteresis, path dependence or history dependence is a property of dynamical systems that have multiple stationary (steady-state) or long-run equilibria. The long-run behaviour of a dynamical system with multiple steady states depends on the initial conditions and on transitory shocks along the adjustment path to the steady state. The example given below has a continuum of steady states, but hysteresis can be present also if there are only a discrete number of steady states.

The natural rate of unemployment is hysteretic if this period's value of the natural rate of unemployment depends on the past history of the actual rate of unemployment. A simple linear process is shown in 11.37 below:

$$\begin{gathered} u_N(t) = \beta u(t-1) + (1-\beta)u_N(t-1) \\ 0 \leq \beta \leq 1 \end{gathered} \tag{4.56}$$

This means that the current value of the natural rate depends on the entire past history of actual rates:

$$u_N(t) = \beta \sum_{i=0}^{T-1} (1-\beta)^i u(t-1-i) + (1-\beta)^T u_N(t-T)$$

There are several theories that support the view that the natural rate may be history-dependent. The first is the *human capital decumulation* explanation of hysteresis in the natural rate. This holds that the experience of actual unemployment hurts both the aptitude for work (loss of skills) and the attitude towards work (demotivation, discouragement). The human capital, that is, the efficiency of the unemployed component of the labour force therefore declines when unemployment is high. This may be especially relevant for the long-term unemployed and for young labour force entrants who miss out on labour market experience during the formative years of their lives. While labour supply in physical units may remain the same, labour supply in units of efficiency or quality falls. The employed become a progressively smaller influence on the wage bargain.

The second explanation is *insider-outsider theory*, due to Gregory [16, Boo], elaborated by Snower and Lindbeck [20] and popularised by Blanchard and Summers [4]. This views the labour force as segmented between insiders and outsiders. The insiders are the employed workers, who negotiate with the employers without taking account of the implications of their wage bargain for the employment prospects of the outsiders. The outsiders are assumed to be completely disenfranchised in the wage and employment negotiations among the insiders. Effective barriers to entry (both by workers and by new firms employing only outsiders) are required for this to be a robust equilibrium. When workers become unemployed, they cease to be insiders and become outsiders. They no longer exercise any restraining influence on the wage and employment negotiations of the remaining insiders. Again, the unemployment cease to represent an effective supply of labour.

An explanation in the same spirit as the human capital theory of hysteresis in the natural rate is the *physical capital stock theory* of hysteresis. It notes that most episodes of disinflation and high unemployment are recessions with low levels of capital formation. If labour and capital are complements in the

production function, this will reduce the demand for labour. The full employment level of employment therefore falls or the natural rate of unemployment rises.

For illustrative purposes, we shall combine 11.37 with a simple augmented Phillips curve, 4.33, and static core inflation, $\bar{\pi}(t) = \pi(t-1)$. This yields

$$\pi(t) = -\alpha\left(u(t) - u_N(t)\right) + \pi(t-1) \tag{4.57}$$

Equations 11.37 and 4.57 imply[2]:

$$\Delta\pi(t) = (1-\beta)\Delta\pi(t-1) - \alpha\Delta u(t) \tag{4.58}$$

Note that the *change* in the inflation rate in period t depends only on the *change* in the unemployment rate and not on the *level* of the unemployment rate. Equation 4.58 implies that, with hysteresis in the unemployment rate, the sacrifice ratio is infinite. Consider the simplest special case where $\beta = 1$, and therefore, the natural rate this period equals last period's actual rate: $u_N(t) = u(t-1)$. In order to reduce the inflation rate in period t by 1%, the unemployment rate in period t has to rise by $\frac{1}{\alpha}$. If the natural rate were constant, the unemployment rate could be reduced back to its original level in period $t+1$, and inflation in periods $t+1$ and beyond would remain at the new 1% lower level, since core inflation in period $t+1$ equals the lower actual rate of inflation in period t. However, because of the hysteresis in the natural rate, $u_N(t+1) = u(t)$ and in order to stop the inflation rate from rising back to its original level, the unemployment rate will have to stay permanently at its new higher level. While the long-run Phillips curve is still vertical, it can be vertical at any level of the unemployment rate, depending on the past history of the unemployment rate. The actual rate of unemployment drags the natural rate of unemployment with it as time passes. The restraining effect on inflation from a higher unemployment rate therefore wears off over time and ultimately disappears altogether (in our simple example it is gone after one period). Hysteresis in the natural rate of unemployment therefore has important implications for the cost-benefit analysis of anti-inflationary policy:

$$\sigma = \infty$$

This approach to hysteresis in the natural rate of unemployment is very similar in spirit to models of wage and price inflation that make the rate of

[2] Δ is the backward difference operator, that is, $\Delta x(t) = x(t) - x(t-1)$

inflation a function of the change in unemployment rather than its level. A simple example with static core inflation is given below:

$$\Delta\pi(t) = -\alpha\Delta u(t)$$

This specification implies that as soon as unemployment stabilizes, no matter at what level, inflation stops rising or falling. The sacrifice ratio is again infinite.

Many empirical Phillips curves that do have the long-run natural rate property nevertheless have the change in the unemployment rate as a determinant of short-run inflation dynamics. An example is the following

$$\pi(t) = \bar{\pi}(t) - \alpha_0\left[u(t) - u_N(t)\right] - \alpha_1\Delta u(t)$$

The natural rate is exogenous in this example. The contribution of $\Delta u(t)$ to the short-run inflation dynamics is sometimes referred to as a 'speed limit effect'.

4.7 More Nominal Rigidities

4.7.1 Menu Costs

4.7.2 Monopolistic Competition

4.7.3 Staggered Wage and Price Setting

Chapter 5

More on the Natural Rate of Unemployment

5.1 Flow Models of the Labour Market

The total population of an economy, $P > 0$, is the sum of the population of working age, $L > 0$ (the labour force) and the non-working age population, $N > 0$. We take the division between working-age population and non-working age population to be a legal or conventional one. Anyone under 16 and over 65 in the UK, for instance, could be classified as non-working age. Of course some non-working age people work. We all know of the teenage newspaper round and of 83 year old house painters, but these exceptions will be ignored in what follows.

$$P \equiv L + N \tag{5.1}$$

The working-age population consists of the active population $A > 0$, and the inactive, $I > 0$:

$$L \equiv A + I \tag{5.2}$$

The active population is either employed, $E > 0$, or unemployed, $U > 0$:

$$A \equiv E + U \tag{5.3a}$$

The size of the total population varies because of births, deaths and net immigration. While it is safe to assume, even in these neo-Victorian times, that all births represent additions to the non-working population, deaths and net immigration can affect E, U and I.

We therefore have four labour market states, employment, unemployment, inactivity and non-working age. The incumbent counts for these four labour market states are, respectively, E, U, I and N. We will treat these incumbent counts as continuous, even though, strictly speaking, they are integers.

The flow approach to the labour market treats the number of people in each of these states as a predetermined state variable. The have the dimension of a stock at a point in time. At any point in time the values of these stocks are therefore inherited from the past and given. Attention is focussed on transitions between these states and on the net additions to the four states due to births, deaths and migration. Transitions and net additions have the dimension of a flow over a period of time. In the continuous time approach adopted in this chapter, they are finite instantaneous rates of flow.

The key underlying idea behind the flow model of the labour market is that all transitions between states, and all net additions, are subject to costs. Theses generalised adjustment or transition costs make it suboptimal to engage in instantaneous, portfolio-style reshuffles between the states. Different approaches emphasize different kinds of costs. When adjustment and transition costs are increasing and strictly convex functions of the speed of the adjustment, optimal adjustments and transitions will be smoothed out over time. When these costs are lumpy, say fixed and sunk costs of leaving one state and entering another, adjustments and transitions will be bunched rather than smoothed over time.

We define the following notation:

$B^i(t) \geq 0$, $i = E, U, I, N$, is the number of births in group i in period t.

$D^i(t) \geq 0$, $i = E, U, I, N$, is the number of deaths in group i in period t.

$M^i(t)$, $i = E, U, I, N$, is the number of net immigrants in group i in period t.

$\pi_{ij}(t),\ i, j = E, U, I, N,\ 0 \leq \pi_{ij} \leq 1$, is the proportion of group j that moves to group i in period t.

When considering the flows between the four labour market states, it is helpful to define a net external additions category, labelled G, to account for births, deaths and net migration. . immigration, that is, for population growth. With this addition, we can represent the gross flows between the various states using the following transition matrix:

Gross flows

		From	From	From	From	From
		G	E	U	I	N
Into	G		D^E	D^U	D^I	D^N
Into	E	$B^E + M^E$	$\pi_{EE}E$	$\pi_{EU}U$	$\pi_{EI}I$	$\pi_{EN}N$
Into	U	$B^U + M^U$	$\pi_{UE}E$	$\pi_{UU}U$	$\pi_{UI}I$	$\pi_{UN}N$
Into	I	$B^I + M^I$	$\pi_{IE}E$	$\pi_{IU}U$	$\pi_{II}I$	$\pi_{IN}N$
Into	N	$B^N + M^N$	$\pi_{NE}E$	$\pi_{NU}U$	$\pi_{NI}I$	$\pi_{NN}N$

(5.4)

From the gross flows we can derive the net changes in the number of occupants for each of the four labour market states as follows:

Net flows

$$\dot{E} \equiv B^E - D^E + M^E + \pi_{EU}U - \pi_{UE}E + \pi_{EI}I - \pi_{IE}E + \pi_{EN}N - \pi_{NE}E \tag{5.5}$$

$$\dot{U} \equiv B^U - D^U + M^U + \pi_{UE}E - \pi_{EU}U + \pi_{UI}I - \pi_{IU}U + \pi_{UN}N - \pi_{NU}U \tag{5.6}$$

$$\dot{I} \equiv B^I - D^I + M^I + \pi_{IE}E - \pi_{EI}I + \pi_{IU}U - \pi_{UI}I + \pi_{IN}N - \pi_{NI}I \tag{5.7}$$

$$\dot{N} \equiv B^N - D^N + M^N + \pi_{NE}E - \pi_{EN}N + \pi_{NU}U - \pi_{UN}N + \pi_{NI}I - \pi_{IN}N \tag{5.8}$$

It follows that

$$\dot{P} \equiv B^E + B^U + B^I + B^N + M^E + M^U + M^I + M^N - (D^E + D^U + D^I + D^N) \tag{5.9}$$

Also the fraction of the occupants of each labour market state that remain in that state is given by:

$$\pi_{EE} \equiv 1 - \pi_{UE} - \pi_{IE} - \pi_{NE} \tag{5.10}$$

$$\pi_{UU} = 1 - \pi_{EU} - \pi_{IU} - \pi_{NU} \tag{5.11}$$

$$\pi_{II} = 1 - \pi_{EI} - \pi_{UI} - \pi_{NI} \tag{5.12}$$

$$\pi_{NN} = 1 - \pi_{EN} - \pi_{UN} - \pi_{NN} \tag{5.13}$$

The gross and net flows are shown in Figure 5.1

Figure 5.1 here

So far, all we have done is list a number of identities. In what follows we will interpret the fractions of the incumbents leaving or entering a state as transition probabilities. We will then try and impose some economic structure on these probabilities, and on the net additions. This will produce a system of differential equations that characterises the behaviour over time of the incumbent counts in the four labour market states and of the total population. The natural rate of unemployment, and a variety of other natural rates, will be the stationary unemployment rate(s) of this dynamic system (if it exists).

For this labour market model to have a well-behaved steady state, behavioural relationships all must satisfy a homogeneity property that permits all stock-flow and stock-stock rations to be constant.

Let $\ell \equiv \frac{L}{P}$, $n \equiv \frac{N}{P}$, $a \equiv \frac{A}{L}$, $i \equiv \frac{I}{L}$, $e \equiv \frac{E}{L}$, $u \equiv \frac{U}{L}$. We call ℓ the labour force participation rate, n the labour force non-participation rate, a the activity rate,i the inactivity rate, e the employment rate and u the unemployment rate.

We also define: $b^E \equiv \frac{B^E}{E}$; $d^E \equiv \frac{D^E}{E}$; $m^E \equiv \frac{M^E}{E}$; $b^U \equiv \frac{B^U}{U}$; $d^U \equiv \frac{D^U}{U}$; $m^U \equiv \frac{M^U}{U}$; $b^I \equiv \frac{B^I}{I}$; $d^I \equiv \frac{D^I}{I}$; $m^I \equiv \frac{M^I}{I}$; $b^N \equiv \frac{B^N}{N}$; $d^I \equiv \frac{D^N}{N}$; $m^N \equiv \frac{M^N}{N}$.

This permits us to rewrite the equations of motion as follows:

$$\dot{e} \equiv \left(b^E - d^E + m^E - (1 - \pi_{EE} + \frac{\dot{L}}{L}) \right) e + \pi_{EU} u + \pi_{EI} i + \pi_{EN} \frac{n}{\ell} \tag{5.14}$$

$$\dot{u} \equiv \left(b^U - d^U + m^U - (1 - \pi_{UU} + \frac{\dot{L}}{L}) \right) u + \pi_{UE} e + \pi_{UI} i + \pi_{UN} \frac{n}{\ell} \quad (5.15)$$

$$\dot{i} \equiv \left(b^I - d^I + m^I - (1 - \pi_{II} + \frac{\dot{L}}{L}) \right) i + \pi_{IU} u + \pi_{IE} e + \pi_{IN} \frac{n}{\ell} \quad (5.16)$$

$$\frac{d\left(\frac{n}{\ell}\right)}{dt} \equiv \left(b^N - d^N + m^N - (1 - \pi_{NN} + \frac{\dot{L}}{L}) \right) \frac{n}{\ell} + \pi_{NE} e + \pi_{NU} u + \pi_{NI} i \quad (5.17)$$

Let ν be the fraction of the total population that is not of working age, that is, $\nu \equiv \frac{N}{P}$. Since $L \equiv E + U + I \equiv P - N$, it follows that

$$\frac{\dot{L}}{L} = \frac{P}{L}\frac{\dot{P}}{P} - (\frac{P}{L} - 1)\frac{\dot{N}}{N} = \frac{\dot{P}}{P} - \left(\frac{\nu}{1-\nu} \right) \frac{\dot{\nu}}{\nu}$$

$$\frac{\dot{P}}{P} \equiv \ell \left((b^E + m^E - d^E)e + (b^U + m^U - d^U)u + (b^I + m^I - d^I)i + (b^N + m^N - d^N)\frac{n}{\ell} \right) \quad (5.18)$$

So far, the only structure we have imposed on the transition probabilities is that all births occur in the non-working population, that is,

$$b^E = b^U = b^I = 0 \quad (5.19)$$

Chapter 6

Some Useful Dynamic AD-AS Models

6.1 Inflation Dynamics in the Closed AD-AS Model

We consider a stripped-down, log-linear version of the IS-LM model with a simple accelerationist Phillips curve. All parameters are positive. All variables except interest rates are natural logarithms. $\bar{y}$ is the exogenous and constant level of capacity output.

$$y = -\gamma r + \theta(m - p) \tag{6.1}$$

$$m - p = ky - \lambda i \tag{6.2}$$

$$\dot{\pi} = \alpha(y - \bar{y}) \tag{6.3}$$

$$r \equiv i - \pi \tag{6.4}$$

$$\pi \equiv \dot{p} \tag{6.5}$$

$$\mu \equiv \dot{m} \tag{6.6}$$

$$\ell \equiv m - p \tag{6.7}$$

This example is due to Tobin and illustrates the conflict between the stabilising effect of a higher price *level* in reducing aggregate demand and the destabilising effect of a higher expected rate of inflation in boosting aggregate demand. Note that both the price level, p, and the rate of inflation, π, are assumed to be predetermined variables.

6.1.1 Nominal Interest Rate Control

Mathnote

Let

$$\dot{x}(t) = a(t)x(t) + z(t) \tag{6.8}$$

x is a predetermined state variable and z is an exogenous or forcing variable. The boundary condition takes the form of an initial condition: $x(t_0) = \bar{x}(t_o)$.

A useful visual way for interpreting the behaviour of a first order differential equation is to plot equation 6.8 in a diagram with $\dot{x}$ on the vertical axis and x on the horizontal axis. For a constant value of z, say $z = \bar{z}$, this looks as follows (note that the steady state solution for x, denoted $\tilde{x}$, which obtains when $\dot{x} = 0$, is given by

$$\tilde{x} = -a^{-1}\bar{z}$$

The solution for 6.8 is

$$x(t) = \bar{x}(t_0)e^{\int_{t_0}^{t} a(v)dv} + \int_{t_0}^{t} e^{\int_{s}^{t} a(v)dv} z(s)ds \tag{6.9}$$

When $a(t)$ is a constant, this simplifies to:

$$x(t) = \bar{x}(t_0)e^{a(t-t_0)} + \int_{t_0}^{t} e^{a(t-s)}z(s)ds \tag{6.10}$$

Check this by differentiating the proposed solutions using Leibniz's rule:

Let

$$F(t) = \int_{\alpha(t)}^{\beta(t)} f(t,s)ds \tag{6.11}$$

then

$$\frac{dF(t)}{dt} = \beta'(t)f\left(t,\beta(t)\right) - \alpha'(t)f\left(t,\alpha(t)\right) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t}f(t,s)ds$$

The authorities use the short nominal interest rate, i, as the monetary instrument. The nominal money stock is endogenous. The model can be written in terms of a single state variable, π :

$$\dot{\pi} = \frac{\alpha\gamma}{1-\theta k}\pi - \frac{\alpha(\gamma+\theta\lambda)}{1-\theta k}i - \alpha\bar{y} \tag{6.12}$$

The steady state of the model ($\dot{\pi} = 0$) is fairly classical:

$$y = \bar{y} \tag{6.13}$$

$$r = -\gamma^{-1}\theta\lambda i - \gamma^{-1}(1-\theta k)\bar{y} \tag{6.14}$$

$$l = k\bar{y} - \lambda i \tag{6.15}$$

$$\pi = \mu = (1+\gamma^{-1}\theta\lambda)i + \gamma^{-1}(1-\theta k)\bar{y} \tag{6.16}$$

The real interest rate depends on the nominal interest rate because a higher nominal interest rate reduces the demand for real money balances and real money balances affect the demand for goods and services through the real balance effect.

Since the real balance effect, θ, is likely to be small, equation 6.12 is unstable, as $\frac{\alpha\gamma}{1-\theta k} > 0$. This amounts to the assumption that the stabilising Pigou or real balance effect is dominated by the destabilising expected inflation or Fisher effect. This instability was analysed at length by Wicksell: with a given nominal interest rate, a higher rate of inflation reduces the real interest rate. This raises the demand for goods and services and further raises inflation. To stabilise the model, the nominal interest rate has to respond to the rate of inflation. A simple nominal interest rate feedback rule popularised by Taylor, can under certain conditions do the job. The Taylor rule is given by:

$$i = \bar{i} + \beta\pi + \eta y \tag{6.17}$$

Since nothing depends on it, we simplify the Taylor rule to depend only on the rate of inflation:

$$i = \bar{i} + \beta\pi \tag{6.18}$$

With the simplified Taylor rule, equation 6.12 becomes

$$\dot{\pi} = \left(\frac{\alpha\gamma(1-\beta) - \theta\alpha\lambda\beta}{1-\theta k}\right)\pi - \frac{\alpha(\gamma + \theta\lambda)}{1-\theta k}\bar{i} - \alpha\bar{y} \tag{6.19}$$

It is clear from 6.19 that, provide the nominal interest rate responds sufficiently vigorously to higher inflation, the model becomes stable. The mechanics of this stabilising feedback rule are very transparent when there is no real balance effect: $\theta = 0$. In this case 6.19 becomes

$$\dot{\pi} = \alpha\gamma(1-\beta)\pi - \alpha\gamma\bar{i} - \alpha\bar{y} \tag{6.20}$$

The model with the Taylor rule is stable if and only if a higher rate of inflation leads to a more than one-for-one increase in the nominal interest rate, so as to raise the real interest rate, dampen aggregated demand and reduce inflation. When $\beta > 1,\ \alpha\gamma(1-\beta) < 0$.

6.1.2 Monetary growth control

Mathnote

A useful reference is Buiter [?]. Consider the following system of 2 first order linear differential equations with constant coefficients:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + B\ z(t) \tag{6.21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

x_1 and x_2 are state variables. They can be either predetermined, in which case the appropriate boundary condition is an initial condition, or non-predetermined, in which case the appropriate boundary condition is a terminal condition, yet to be determined. z is an exogenous or forcing variable.

Let ρ_1 and ρ_2 be the two eigenvalues of A. They are found by solving the characteristic equation

$$|A - \rho I| = 0$$

where I is the identity matrix,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the 2×2 case, the eigenvalues are given by

$$\rho_{1,2} = \frac{1}{2}\left[a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}\right]$$

Let $R = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}$

The eigenvalues can be distinct or repeated, real, if $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \geq 0$ or complex conjugate, if $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) < 0$. When roots are complex, they come in pairs, $\rho_1 = \alpha + \beta i$ and $\rho_2 = \alpha - \beta i$, where $i = \sqrt{-1}$. α is called the real part and βi the imaginary part of the pair of complex conjugate roots.

The following trigonometric identities show how complex roots are associated with cyclical behaviour. A_1 and A_2 are arbitrary constants. When ρ_1 and ρ_2 are complex conjugate roots, the arbitrary constants will also have to be complex conjugate numbers if the solution is going to be real.

$$\begin{aligned}
&A_1 e^{(\alpha+\beta i)t} + A_2 e^{(\alpha-\beta i)t} \\
&= e^{\alpha t}\left(A_1 e^{\beta i t} + A_2 e^{-\beta i t}\right) \\
&= e^{\alpha t}\left(A_1(\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)\right) \\
&= e^{\alpha t}\left((A_1 + A_2)\cos\beta t + i(A_1 - A_2)\sin\beta t)\right) \\
&= e^{\alpha t}\left(B_1\cos\beta t + B_2\sin\beta t)\right)
\end{aligned} \tag{6.22}$$

where B_1 and B_2 are new arbitrary constants defined by

$$B_1 = A_1 + A_2$$

and

$$B_2 = i(A_1 - A_2)$$

We can write the expression in 6.22 slightly differently as follows:

Let

$$B_1 = B\cos\epsilon$$

and

$$B_2 = B\sin\epsilon$$

then

$$\begin{aligned} &e^{\alpha t}\left(B_1\cos\beta t + B_2\sin\beta t)\right) = e^{\alpha t}\left(B\cos\epsilon\cos\beta t + B\sin\epsilon\sin\beta t)\right) \\ &= Be^{\alpha t}\cos(\beta t - \epsilon) \end{aligned}$$

The expression $\cos(\beta t - \epsilon)$ is a periodic function with values between -1 and $+1$. The period of the cycle (the duration of a complete cycle) is $\frac{2\pi}{\beta}$ and the amplitude is 1. The frequency of the cycle is the reciprocal of the period, $\beta/2\pi$.

We denote the trace of the matrix A by $TR(A)$ and the determinant of A by $\Delta(A)$.

$$TR(A) \equiv a_{11} + a_{22}$$

$$\Delta(A) \equiv a_{11}a_{22} - a_{12}a_{21}$$

It follows that

$$TR(A) = \rho_1 + \rho_2$$

and

$$\Delta(A) = \rho_1\rho_2$$

We assume that A can be diagonalised by a similarity transformation, that is that

$$A = VRV^{-1} \tag{6.23}$$

V is a matrix (in our example 2×2) whose columns are the (right) eigenvectors of A,

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

This will be true if and only if V is of full rank. In our case this means that there are 2 linearly independent eigenvectors. For this to hold it is sufficient that all eigenvalues be distinct.

The eigenvectors are the solutions to[1]

$$Av_i = \rho_i v_i \qquad\qquad i = 1, 2$$

In the 2×2 case, the eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ \frac{\rho_1 - a_{11}}{a_{12}} \end{bmatrix}$$

or

$$v_1 = \begin{bmatrix} 1 \\ \frac{a_{21}}{\rho_1 - a_{22}} \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} 1 \\ \frac{\rho_2 - a_{11}}{a_{12}} \end{bmatrix}$$

or

[1] Note that the eigenvectors are unique only upto a non-zero multiplicative constant.

$$v_2 = \begin{bmatrix} 1 \\ \frac{a_{21}}{\rho_2 - a_{22}} \end{bmatrix}$$

We also define:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = V^{-1}x \tag{6.24}$$

and

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} = V^{-1} \tag{6.25}$$

We can now rewrite 6.21 as follows:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + WBz$$

or

$$\dot{y}_1 = \rho_1 y_1 + \eta_1 \tag{6.26}$$

$$\dot{y}_2 = \rho_2 y_2 + \eta_2 \tag{6.27}$$

where

$$\eta_1 = (w_{11}b_1 + w_{12}b_2)\, z$$

and

$$\eta_2 = (w_{21}b_1 + w_{22}b_2)\, z$$

(A) Two predetermined state variables

In this case the two boundary conditions take the form of initial conditions

$$x_1(t_0) = \bar{x}_1(t_o)$$

$$x_2(t_0) = \bar{x}_2(t_o)$$

Equations 6.26 and 6.27 can now be solved as follows:

The solutions are

$$y_1(t) = y_1(t_0)e^{\rho_1(t-t_0)} + \int_{t_0}^{t} e^{\rho_1(t-s)}\eta_1(s)ds \tag{6.28}$$

$$y_2(t) = y_2(t_0)e^{\rho_2(t-t_0)} + \int_{t_0}^{t} e^{\rho_2(t-s)}\eta_2(s)ds \tag{6.29}$$

The initial conditions for y_1 and y_2 are found from equations 6.24 and 6.25, and the initial conditions for x_1 and x_2 :

$$\begin{bmatrix} y_1(t_0) \\ y_2(t_{0)} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t_0) \\ \bar{x}_2(t_0) \end{bmatrix} \tag{6.30}$$

Having solved for y_1 and y_2, we can recover x_1 and x_2 for $t \geq t_0$ from

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \tag{6.31}$$

The reason why this transformation to canonical variables is useful, is that y_1 and y_2 depend each on one of the eigenvalues only. The stability of the homogenous equation can therefore be seen clearly to require that both eigenvalues have negative real parts.

Consider the case where the exogenous variables are constant. The *isoclines* (the loci of combinations of x_1 and x_2 for which $\dot{x}_1 = 0$, respectively $\dot{x}_2 = 0$) are given by the following 2 equations:

The $\dot{x}_1 = 0$ locus:

$$x_2 = a_{12}^{-1} b_1 z - a_{12}^{-1} a_{11} x_1 \tag{6.32}$$

The $\dot{x}_2 = 0$ locus:

$$x_2 = a_{22}^{-1} b_2 z - a_{22}^{-1} a_{21} x_1 \tag{6.33}$$

For constant values of the exogenous variables, the steady state values of x_1 and x_2, $\tilde{x}_1$ and $\tilde{x}_2$ are found by setting $\dot{x}_1 = \dot{x}_2 = 0$. This implies

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = -A^{-1} B z \tag{6.34}$$

We can graphically represent the dynamics of this system by drawing the two isoclines in x_2, x_1 space. Such a phase diagram can haver several qualitatively different configurations.

(1) Two stable roots, both real.

This will be the case if $\Delta(A) > 0,\ TR(A) < 0$ and $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \geq 0$.

In this case the system will converge monotonically to the steady state (there will be no oscillations).

(2)Two stable complex conjugate roots.

This will be the case if $\Delta(A) > 0,\ TR(A) < 0$ and $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) < 0$.

In this case the system will converge to the steady state in an oscillating manner.

(3) One stable root and one unstable root.

This will be the case if $\Delta(A) < 0$. Note that both roots are real in this case.

The system will diverge from the steady state, unless the initial conditions place it on the 'saddle-path'. The equilibrium of this system is called a saddlepoint.

(4) Two unstable roots, both real.

This will be the case if $\Delta(A) > 0,\ TR(A) > 0$ and $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \geq 0$.

In this case the system diverges from the steady state in a monotone manner.

(5)Two unstable complex conjugate roots.

This will be the case if $\Delta(A) > 0,\ TR(A) > 0$ and $(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) < 0$.

In this case the system diverges from the steady state in an oscillating manner.

From 6.24 and 6.25 we can write x_2 as a function of x_1 and y_2

$$x_2 = -w_{22}^{-1}w_{21}x_1 + w_{22}^{-1}y_2 \tag{6.35}$$

or, equivalently,

$$\begin{aligned} x_2 &= v_{21}v_{11}^{-1}x_1 + (v_{22} - v_{11}^{-1}v_{12})y_2 \\ &= v_{21}v_{11}^{-1}x_1 + w_{22}^{-1}y_2 \end{aligned} \tag{6.36}$$

In the $2x2$ case this can be written as

$$x_2 = \left(\frac{\rho_1 - a_{11}}{a_{12}}\right)x_1 + w_{22}^{-1}y_2 \tag{6.37}$$

or, equivalently,

$$x_2 = \left(\frac{a_{21}}{\rho_1 - a_{22}}\right)x_1 + w_{22}^{-1}y_2 \tag{6.38}$$

From 6.24 and 6.25 we can also write x_1 as a function of x_2 and y_1

$$x_1 = -w_{11}^{-1}w_{12}x_2 + w_{11}^{-1}y_1 \tag{6.39}$$

or, equivalently,

$$\begin{aligned} x_1 &= v_{12}v_{22}^{-1}x_2 + (v_{11} - v_{12}v_{22}^{-1}v_{21})y_1 \\ &= v_{12}v_{22}^{-1}x_2 + w_{11}^{-1}y_2 \end{aligned} \tag{6.40}$$

In the $2x2$ case this can be written as

$$x_1 = \left(\frac{\rho_2 - a_{22}}{a_{21}}\right) x_2 + w_{11}^{-1}y_1 \tag{6.41}$$

or, equivalently,

$$x_1 = \left(\frac{a_{12}}{\rho_2 - a_{11}}\right) x_2 + w_{22}^{-1}y_2 \tag{6.42}$$

(B) One predetermined and one non-predetermined state variable

Let x_1 be the predetermined state variable and x_2 the non-predetermined one.

One boundary condition takes the form of an initial condition for the predetermined state variable

$$x_1(t_0) = \bar{x}_1(t_0) \tag{6.43}$$

The second boundary condition is:

$$\text{Choose a continuously convergent solution} \tag{6.44a}$$

We assume that one of the two characteristic roots, ρ_1 is stable and that the other, ρ_2 is unstable, that is,

$$\begin{aligned} \rho_1 &< 0 \\ \rho_2 &> 0 \end{aligned}$$

We solve 6.27 forward. This yields

$$y_2(t) = -\int_t^\infty e^{\rho_2(t-s)}\eta_2(s)ds + ke^{\rho_2 t}$$

where k is an arbitrary constant. Since $\rho_2 > 0$, boundary condition 6.44a implies $k = 0$. This implies:

$$y_2(t) = -\int_t^\infty e^{\rho_2(t-s)}\eta_2(s)ds \tag{6.45}$$

Equations 6.45 and 6.35 imply

$$x_2(t) = -w_{22}^{-1}w_{21}x_1(t) - w_{22}^{-1}\int_t^\infty e^{\rho_2(t-s)}\eta_2(s)ds \tag{6.46}$$

or, equivalently,

$$x_2(t) = v_{21}v_{11}^{-1}x_1(t) - w_{22}^{-1}\int_t^\infty e^{\rho_2(t-s)}\eta_2(s)ds \tag{6.47}$$

In the $2x2$ case this can be written as

$$x_2(t) = \left(\frac{\rho_1 - a_{11}}{a_{12}}\right)x_1(t) - w_{22}^{-1}\int_t^\infty e^{\rho_2(t-s)}\eta_2(s)ds \tag{6.48}$$

or, equivalently,

$$x_2(t) = \left(\frac{a_{21}}{\rho_1 - a_{22}}\right)x_1(t) - w_{22}^{-1}\rho_2^{-1}\eta_2 \tag{6.49}$$

In the case where η_2 is constant, the last four equations can be rewritten as

$$x_2(t) = -w_{22}^{-1}w_{21}x_1(t) - w_{22}^{-1}\rho_2^{-1}\eta_2 \tag{6.50}$$

or, equivalently,

$$x_2(t) = v_{21}v_{11}^{-1}x_1(t) - w_{22}^{-1}\rho_2^{-1}\eta_2 \tag{6.51}$$

In the $2x2$ case this can be written as$(a_{11} - a_{12}w_{22}^{-1}w_{21})$

$$x_2(t) = \left(\frac{\rho_1 - a_{11}}{a_{12}}\right)x_1(t) - w_{22}^{-1}\rho_2^{-1}\eta_2 \tag{6.52}$$

or, equivalently,

$$x_2(t) = \left(\frac{a_{21}}{\rho_1 - a_{22}}\right)x_1(t) - w_{22}^{-1}\ \rho_2^{-1}\eta_2 \tag{6.53}$$

This relationship between the non-predetermined state variable x_2 and the predetermined state variable x_1 is the 'stable manifold' or 'saddlepath'. Since $\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1z$, we know that

$$\begin{aligned}\dot{x}_1(t) &= (a_{11} - a_{12}w_{22}^{-1}w_{21})x_1(t) - a_{12}w_{22}^{-1}\int_t^\infty e^{\rho_2(t-u)}\eta_2(u)du + b_1z(t)\\ &= \rho_1x_1(t) - a_{12}w_{22}^{-1}\int_t^\infty e^{\rho_2(t-u)}\eta_2(u)du + b_1z(t)\end{aligned}$$

It follows that

$$\begin{aligned}x_1(t) &= x\,(t_0)\,e^{(a_{11}-a_{12}w_{22}^{-1}w_{21})(t-t_0)}\\ &+\int_{t_0}^t e^{(a_{11}-a_{12}w_{22}^{-1}w_{21})(t-s)}[b_1z(s) - a_{12}w_{22}^{-1}\int_s^\infty e^{\rho_2(t-u)}\eta_2(u)du]ds\end{aligned}$$

The authorities are assumed to treat the instantaneous rate of growth of the nominal money stock, μ, as the policy instrument. A convenient choice of state variables are π and ℓ, the stock of real money balances. We can write the state-space representation of the model as follows:

$$\begin{bmatrix} \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \begin{bmatrix} \frac{\alpha\gamma}{1+\gamma k\lambda^{-1}} & \frac{\alpha(\theta+\gamma\lambda^{-1})}{1+\gamma k\lambda^{-1}} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \pi \\ \ell \end{bmatrix} + \begin{bmatrix} 0 & -\alpha \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \bar{y} \end{bmatrix} \tag{6.54}$$

The determinant of the matrix $\begin{bmatrix} \frac{\alpha\gamma}{1+\gamma k\lambda^{-1}} & \frac{\alpha(\theta+\gamma\lambda^{-1})}{1+\gamma k\lambda^{-1}} \\ -1 & 0 \end{bmatrix} \equiv A$, denoted $\Delta(A)$, is given by

$$\Delta(A) = \frac{\alpha(\theta + \gamma\lambda^{-1})}{1 + \gamma k\lambda^{-1}} > 0$$

The trace of the same matrix (the sum of its diagonal elements) denoted $TR(A)$ is given by

$$TR(A) = \frac{\alpha\gamma}{1 + \gamma k\lambda^{-1}} > 0$$

Let ρ_1 and ρ_2 be the two characteristic roots or eigenvalues of the matrix A. Since

$$\Delta(A) = \rho_1\rho_2$$

and

$$TR(A) = \rho_1 + \rho_2$$

it follows that both eigenvalues have positive real parts. The system is unstable.

Now consider a Taylor-type rule for the growth rate of the nominal money stock:

$$\mu = \overline{\mu} + \beta\pi \tag{6.55}$$

With this feedback rule, the dynamic system becomes

$$\begin{bmatrix} \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \begin{bmatrix} \frac{\alpha\gamma}{1+\gamma k\lambda^{-1}} & \frac{\alpha(\theta+\gamma\lambda^{-1})}{1+\gamma k\lambda^{-1}} \\ \beta - 1 & 0 \end{bmatrix} \begin{bmatrix} \pi \\ \ell \end{bmatrix} + \begin{bmatrix} 0 & -\alpha \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\mu} \\ \bar{y} \end{bmatrix} \tag{6.56}$$

Note that we cannot stabilise the model with this feedback rule. $TR(A)$ continues to be positive. As long as the $\Delta(A)$ is positive (which will be the case as long as $\beta < 1$), both eigenvalues of A will continue to be positive. We can make $\Delta(A)$ negative, by choosing $\beta > 1$. This means that, when the inflation rate increases, the growth rate of the nominal money stock has to increase more than one-for-one. Thus, when the inflation rate increases, the growth rate of the nominal money stock is increased by so much that the real money stock starts rising at a faster rate. That hardly sounds stabilising, and it isn't, because with $\Delta(A) < 0$, we have one negative and one positive eigenvalue, and the system is still unstable. We can stabilise the system by making μ a function of y. Even simpler is a rule that makes μ a function of ℓ. Consider e.g. the following rule

$$\mu = \overline{\mu} + \zeta\ell \tag{6.57}$$

With this rule, the dynamic system becomes

$$\begin{bmatrix} \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \begin{bmatrix} \frac{\alpha\gamma}{1+\gamma k\lambda^{-1}} & \frac{\alpha(\theta+\gamma\lambda^{-1})}{1+\gamma k\lambda^{-1}} \\ -1 & \zeta \end{bmatrix} \begin{bmatrix} \pi \\ \ell \end{bmatrix} + \begin{bmatrix} 0 & -\alpha \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \overline{\mu} \\ \bar{y} \end{bmatrix} \tag{6.58}$$

It follows that now,

$$\Delta(A) = \frac{\alpha\left(\gamma\zeta + (\theta + \gamma\lambda^{-1})\right)}{1 + \gamma k\lambda^{-1}}$$

$$TR(A) = \frac{\alpha\gamma + \zeta(1 + \gamma k\lambda^{-1})}{1 + \gamma k\lambda^{-1}}$$

For the system to be stable, we require $\Delta(A) > 0$ and $TR(A) < 0$. Provided ζ is sufficiently negative (that is, a higher value of the stock of real money balances leads the authorities to reduce the growth rate of the nominal money stock by a large enough amount, $TR(A)$ can be negative. ζ cannot be too negative, however, or the determinant condition will be violated. Provided $-\frac{\alpha\gamma}{1+\gamma k\lambda^{-1}} > \zeta > -\frac{(\theta+\gamma\lambda^{-1})}{\gamma}$, the system can be stabilised. Note that without feedback control, the system will be unstable both under a nominal interest rate rule and under a monetary growth rule.

Exercise 6.1.1 *Check under what conditions a monetary rule of the following kind, $\mu = \overline{\mu} + \chi y$, will stabilise the system. Interpret your findings.*

6.2 Public Debt Dynamics: the Government Budget Constraint

6.2.1 The Fixed Price Case

We consider a slightly simplified version of the IS-LM model. Physical capital is omitted (see Blinder and Solow [7], [11] and Tobin and Buiter [27]). Disposable income now explicitly include interest income on the public debt. Taxes are assumed to be proportional to output. Expected inflation is exogenous and can be set equal to zero.

$$c(Y_d, i - \hat{\pi}, \frac{M+B}{P}) + \iota(i - \hat{\pi}, Y) + G = Y$$

$$h(i, Y, \frac{M+B}{P}) = \frac{M}{P}$$

$$Y_d \equiv Y + i\frac{B}{P} - T$$

$$T = \tau Y$$

$$P = \bar{P} = 1$$

$$\hat{\pi} = 0$$

$$\dot{M} + \dot{B} \equiv P(G + i\frac{B}{P} - T) \tag{6.59}$$

Equation 6.59 is the government budget identity *aka* the government budget constraint. It says that the financial deficit or budget deficit of the government has to be financed either by printing money or by borrowing. We summarise the model as follows:

$$c(Y(1-\tau)+iB,i,M+B)+\iota(i,Y)+G=Y$$

$$h(i,Y,M+B)=M$$

$$\dot{M}+\dot{B}\equiv G+iB-\tau Y \tag{6.60}$$

Pure Money Financing of Government Deficits

In this case $\dot{B}=0$.

We solve the IS and LM curves for i and Y as functions of the predetermined state variable, M, and the exogenous variables B,G and τ.

This yields

$$i=H(M;B,G,\tau)$$

and

$$Y=F(M;B,G,\tau)$$

with

$$F_M=\frac{\frac{-c_A}{P}h_i-(c_r+\iota_r)\frac{(1-h_A)}{P}}{D}\geq 0$$

$$H_M=\frac{[(1-\tau)c_{Y_d}+\iota_Y-1]\left(\frac{(1-h_A)}{P}\right)+h_Y\frac{c_A}{P}}{D}\lesseqgtr 0$$

For concreteness, we assume that the rightward shift of the LM curve dominates the rightward shift of the IS curve, so $H_M<0$.

$$F_B=\frac{\frac{-c_A}{P}h_i+(c_r+\iota_r)\frac{h_A}{P}}{D}\lesseqgtr 0$$

$$H_B = \frac{\left[1 - ((1-\tau)c_{Y_d} + \iota_Y)\right] \frac{h_A}{P} + h_Y \frac{c_A}{P}}{D} > 0$$

$$F_G = \frac{-h_i}{D} \geq 0$$

$$H_G = \frac{h_Y}{D} > 0$$

$$F_\tau = \frac{Y c_{Y_d} h_i}{D} \leq 0$$

$$H_\tau = \frac{-Y c_{Y_d} h_Y}{D} < 0$$

$$D = ((1-\tau)c_{Y_d} + \iota_Y - 1)\, h_i - h_Y(c_r + \iota_r) > 0$$

Substituting the IS-LM solutions for i and Y into the government budget identity yields

$$\dot{M} \equiv G + H(M; B, G, \tau)B - \tau F(M; B, G, \tau) \tag{6.61}$$

This is a first-order but unfortunately non-linear differential equation in M.

The steady-state solution for M, denoted $\tilde{M}$, obtained by setting $\dot{M} = 0$, is found from

$$G + iB - \tau Y = 0$$

or

$$G + H(\tilde{M}; B, G, \tau)B - \tau F(\tilde{M}; B, G, \tau) = 0$$

This implies

$$\tilde{M} = V(B; G, \tau)$$

with

$$V_B = \frac{-(i + BH_B - \tau F_B)}{\Phi}$$

$$V_G = \frac{-(1 + BH_G - \tau F_G)}{\Phi} > 0 \quad \text{(probably)} \tag{6.62}$$

$$V_\tau = \frac{Y - BH_\tau + \tau F_\tau)}{\Phi} < 0 \quad \text{(probably)}$$

$$\Phi = BH_M - \tau F_M < 0$$

The effect of a government spending increase on the steady state money stock will be positive unless the impact effect of the higher spending is to reduce the budget deficit. This can only happen if the initial level of output rises by so much that it more than offsets the budget deficit-increasing effect of the higher level of public spending itself, as well as the budget deficit-increasing effect of the higher interest rate associated with the higher level of spending. We assume that the impact effect of higher public spending is a larger deficit. Since the deficit is, by assumption, financed by printing money, the long-run money stock increases.

We linearise 8.27 at the steady state. $\tilde{B}$, $\tilde{G}$ and $\tilde{\tau}$ are the steady-state values of the exogenous variables.

This implies

$$\begin{aligned} \dot{M} \approx\ & (BH_M - \tau F_M)(M - \tilde{M}) + (1 + BH_B - \tau F_B)(B - \tilde{B}) \\ & + (1 + BH_G - \tau F_G)(G - \tilde{G}) - (Y - BH_\tau + \tau F_\tau)(\tau - \tilde{\tau}) \end{aligned}$$

The system will be locally stable under monetary financing if and only if $BH_M - \tau F_M < 0$, that is if, ceteris paribus, monetary issuance lowers the budget deficit. This is indeed the case. Other things being equal, a larger stock of money reduces the budget deficit, both by lowering the interest rate and by raising output.

Pure Bond Financing of Government Deficits

We now reverse the roles of money and bonds, with the money stock becoming exogenous and the stock of government interest-bearing debt becoming the predetermined state variable used for financing the deficit. From the government budget identity and the IS-LM solutions for output and the interest rate we obtain:

$$\dot{B} \equiv G + H(B; M, G, \tau)B - \tau F(B; M, G, \tau) \tag{6.63}$$

The steady-state value of the stock of bonds is found from

$$G + H(\tilde{B}; M, G, \tau)\tilde{B} - \tau F(\tilde{B}; M, G, \tau) = 0$$

This implies

$$\tilde{B} = U(M, G, \tau)$$

with

$$U_M = -D^{-1}(BH_M - \tau F_M)$$

$$U_G = -D^{-1}(1 + BH_G - \tau F_G) \tag{6.64}$$

$$U_\tau = D^{-1}(Y - BH_\tau + \tau F_\tau)$$

$$D = i + BH_B - \tau F_B \tag{6.65}$$

Note that D will be positive if, ceteris paribus, a larger stock of interest-bearing debt increases the deficit. This seems likely. A larger stock of bonds directly increases the interest bill. It also raises the interest rate (the IS curve shifts to the right and the LM curve shifts to the left). This 'interest bill' effect can be reversed only if a larger stock of debt reduces the government's primary (non-interest) deficit. This requires that tax revenues go up when the stock of bonds increases. For a given value of τ, this requires a sufficiently

strong positive effect from a larger stock of bonds on output. The effect of a larger stock of bonds on output is a-priori ambiguous. Even if it is positive, it is likely to be small. It therefore seems likely that $D > 0$. If the impact effect of an increase in government spending on the deficit is positive, $1 + BH_G - \tau F_G > 0$, it follows from 6.64 that the steady-state effect of an increase in public spending on the stock of bonds is negative. This seems at odds with the fact that the impact effect of a larger stock of bonds is to increase the deficit, and that subsequent bond issues to finance the increase deficit will, if D in 6.65 is positive, further add to the deficit. The long-run comparative statics are impeccable, but the model is clearly unstable. This is confirmed by linearising 6.63 at the steady state. This yields

$$\begin{aligned}\dot{B} \approx\ & (i + BH_B - \tau F_B)(B - \tilde{B}) + B(H_M - \tau F_M)(M - \tilde{M}) \\ & +(1 + BH_G - \tau F_G)(G - \tilde{G}) + (BH_\tau - \tau F_\tau - Y)(\tau - \tilde{\tau})\end{aligned}$$

This will be locally stable if and only if $(i + BH_B - \tau F_B) = D < 0$. We concluded earlier that $D > 0$.

A Nominal Interest Rate Rule

When the authorities peg the nominal interest rate, the composition of the government's stock of financial liabilities, $\pounds \equiv M + B$, becomes endogenous. We now solve the IS-LM equilibrium conditions for Y and M as functions of the predetermined state variable $\pounds$, and the exogenous variables, i, G and τ. The linearized IS-LM equations are

$$\begin{aligned}&\begin{bmatrix} (1-\tau)c_{Y_d+\iota_y} - 1 & -ic_{Y_d} \\ h_Y & -1 \end{bmatrix} \begin{bmatrix} dY \\ dM \end{bmatrix} \\ &= \begin{bmatrix} -c_A & -(c_r + \iota_r + Bc_{Y_d}) & -1 & Yc_{Y_c} \\ -h_A & -h_i & 0 & 0 \end{bmatrix} \begin{bmatrix} d\pounds \\ di \\ dG \\ d\tau \end{bmatrix}\end{aligned} \tag{6.66}$$

Let

$$Y = F(\pounds; i, G, \tau)$$

$$M = J(\pounds; i, G, \tau)$$

It follows from 6.66 that

$$F_{£} = D^{-1}(c_A - h_A i c_{Y_d}) \gtreqless 0$$

we assume

$$F_{£} > 0$$

The LM curve is horizontal because the authorities peg the nominal interest rate. The wealth effect of an increase in financial wealth shifts the IS curve to the right. However, the equilibrium money stock increases, because of the wealth effect on the demand for money. It is therefore possible that the stock of interest-bearing debt, B, falls, lowering disposable income and reducing consumer demand. We ignore this possibility.

$$J_{£} = D^{-1}\left[\left(1 - (1-\tau)c_{Y_d} - \iota_Y\right)h_A + h_y c_A\right] > 0$$

It is likely, since $0 < h_A < 1$, and c_A is small, that

$$0 < J_{£} < 1$$

The equilibrium money stock increases when the total stock of financial wealth increases, but less than one-for-one.

$$F_i = D^{-1}\left(c_r + \iota_r + Bc_{Y_d} - ic_{Y_d}h_i\right) \gtreqless 0$$

We assume

$$F_i < 0$$

Again there may be offsetting effects through the effect of a higher interest rate on disposable income, both directly, through the effect of a higher interest rate on interest income for a given stock of public interest-bearing debt, and indirectly because a higher interest rate reduces the equilibrium money stock (assuming a higher interest rate does not boost the demand for goods and services), thus increasing B for a given stock of $£$.

$$J_i = D^{-1}\left[\left(1 - (1-\tau)c_{Y_d} - \iota_Y\right)h_i + h_A(c_r + \iota_r + Bc_{Y_d})\right] \gtreqless 0$$

We assume

$$J_i < 0$$

$$F_G = D^{-1} > 0$$

$$J_G = D^{-1} h_Y > 0$$

$$F_\tau = -D^{-1} Y c_{Y_d} < 0$$

$$J_\tau = -D^{-1} h_Y Y c_{Y_d} < 0$$

$$D = 1 - (1-\tau) c_{Y_d} - \iota_Y + i c_{Y_d} h_Y > 0$$

The steady state stock of government liabilities, $\tilde{\pounds}$ is determined from

$$G + i\left(\tilde{\pounds} - J(\tilde{\pounds}; i, G, \tau)\right) - \tau F(\tilde{\pounds}; i, G, \tau) = 0$$

This implies

$$\pounds = N(i, G, \tau)$$

$$N_i = D^{-1}\left(iJ_i + \tau F_i - B\right)$$

$$N_G = D^{-1}\left(iJ_G + \tau F_G - 1\right)$$

$$N_\tau = D^{-1}\left(iJ_\tau + \tau F_\tau + Y\right)$$

$$D = i(1 - J_{\pounds}) - \tau F_{\pounds} \tag{6.67}$$

Substituting the IS-LM solutions for Y and M into the government budget identity yields

$$\dot{\pounds} \equiv G + i\left(\pounds - J(\pounds; i, G, \tau)\right) - \tau F\left(\pounds; i, G, \tau\right)$$

Taking a linear approximation at the steady state yields

$$\begin{aligned}\dot{\pounds} \approx\ & \left(i(1 - J_{\pounds}) - \tau F_{\pounds}\right)(\pounds - \tilde{\pounds}) + \left(B - iJ_i - \tau F_i\right)(i - \tilde{i}) \\ & + (1 - iJ_G - \tau F_G)(G - \tilde{G}) - (Y + iJ_\tau + \tau F_\tau)(\tau - \tilde{\tau})\end{aligned}$$

The system will be locally stable if and only if $i(1 - J_{\pounds}) - \tau F_{\pounds} < 0$. This will be the case only if a larger stock of government financial liabilities has a strong positive effect on output (the rightward shift of the IS curve due to the wealth effect on consumption is very strong, raising output and tax revenues by enough to reduce the deficit, despite the likely increase in the government's interest bill associated with an increase in $\pounds$).

The likely result is that, with an exogenous nominal interest rate, the deficit-debt dynamics will be unstable.

Exercise 6.2.1 *Find a feedback rule for the nominal interest rate that will stabilise the model. Explain how this rule achieves its objective.*

6.3 Public debt dynamics with endogenous inflation

We add an accelerationist Phillips curve to the IS-LM model and the government budget identity. We assume rational inflation expectations, so $\hat{\pi} = \pi$.

$$c(Y(1-\tau) + i\frac{B}{P}, i - \pi, \frac{M+B}{P}) + \iota(i - \pi, Y) + G = Y$$

$$h(i, Y, \frac{M+B}{P}) = \frac{M}{P}$$

$$\begin{aligned} \dot{\pi} &= \alpha(\frac{Y}{\bar{Y}} - 1) \\ \alpha &> 0 \end{aligned}$$

$$\dot{M} + \dot{B} \equiv P(G + i\frac{B}{P} - \tau Y)$$

$$\pi = \frac{\dot{P}}{P}$$

The level of capacity output, $\bar{Y}$, grows at the exogenous proportional rate n, that is,

$$\frac{d\bar{Y}}{dt} = n\bar{Y}$$

To be able to consider steady states with non-zero growth rates of all real variables, we assume that the consumption function is linear homogeneous in Y_d and A, that the investment function is linear homogeneous in Y and that the money demand function is linear homogeneous in Y and A. We can therefore rewrite the model as follows:

$$c\left((1-\tau)\frac{Y}{\bar{Y}} + i\frac{B}{P\bar{Y}}, i - \pi, \frac{M+B}{P\bar{Y}}\right) + \iota(i - \pi, \frac{Y}{\bar{Y}}) + \bar{g} = \frac{Y}{\bar{Y}} \tag{6.68}$$

$$h(i, \frac{Y}{\bar{Y}}, \frac{M+B}{\bar{Y}P}) = \frac{M}{P\bar{Y}}$$

$$\dot{\pi} = \alpha(\frac{Y}{\bar{Y}} - 1)$$

$$\frac{\dot{M} + \dot{B}}{P\bar{Y}} \equiv \bar{g} + i\frac{B}{P\bar{Y}} - \tau\frac{Y}{\bar{Y}}$$

$$\pi = \frac{\dot{P}}{P}$$

where $\bar{g} \equiv \frac{G}{\bar{Y}}$.

We define

$$\bar{m} \equiv \frac{M}{P\bar{Y}}$$

$$\bar{b} \equiv \frac{B}{P\bar{Y}}$$

and

$$\bar{y} \equiv \frac{Y}{\bar{Y}}$$

This permits us to rewrite the model as

$$c\left((1-\tau)\bar{y} + i\bar{b}, i-\pi, \bar{m}+\bar{b}\right) + \iota(i-\pi, \bar{y}) + \bar{g} = \bar{y} \tag{6.69}$$

$$h(i, \bar{y}, \bar{m}+\bar{b}) = \bar{m} \tag{6.70}$$

$$\dot{\pi} = \alpha(\bar{y} - 1) \tag{6.71}$$

$$\frac{d(\bar{m}+\bar{b})}{dt} \equiv \bar{g} + i\bar{b} - \tau\bar{y} - (\pi+n)(\bar{m}+\bar{b}) \tag{6.72}$$

6.3.1 Steady State

In steady state, the rate of inflation is constant and the ratio of real money to trend GDP and of real bonds to trend GDP are constant ($\dot{\pi} = \frac{d\bar{m}}{dt} = \frac{d\bar{b}}{dt} = 0$).

This implies

$$\bar{y} = 1 \tag{6.73}$$

$$c\left(1-\tau+i\bar{b}, i-\pi, \bar{m}+\bar{b}\right) + \iota(i-\pi, 1) + \bar{g} = 1 \tag{6.74}$$

$$h(i, 1, \bar{m} + \bar{b}) = \bar{m} \tag{6.75}$$

$$(\pi + n)(\bar{m} + \bar{b}) \equiv \bar{g} + i\bar{b} - \tau \tag{6.76}$$

Note that the government budget need not be balanced in the long run. Steady state requires that the ratio of money to nominal GDP and the ratio of bonds to nominal GDP be constant. For instance, with a value of $\bar{m} + \bar{b}$ of 0.6 (the Maastricht ceiling for the ratio of general government debt to annual GDP), an inflation rate of 0.025 per annum and a real growth rate of 0.025 per annum, the government deficit as a fraction of GDP would be 0.030 or three percent of GDP.

6.3.2 A Constant Interest-Bearing Public Debt-GDP Ratio and Endogenous Money

Assume that the government issues or retires nominal interest-bearing debt in the exact amount required to stabilise the interest-bearing debt-GDP ratio at $\bar{b} = \bar{\bar{b}}$. Note that this means that

$$\dot{B} = (\pi + n)B$$

We assume that $\bar{g}$ and τ are constant. The interest rate and the change in the nominal money stock are therefore endogenously determined.

We solve the IS and LM equations 6.69 and 6.70 for $\bar{y}$ and i as functions of $\bar{m}$, $\bar{b}$, π, g and τ.

$$\bar{y} = F(\bar{m}, \bar{b}, \pi, g, \tau)$$

with

$$F_{\bar{m}} > 0; F_{\bar{b}} \lesseqgtr 0; F_{\pi} > 0; F_{\bar{g}} > 0; F_{\tau} < 0$$

$$i = H(\bar{m}, \bar{b}, \pi, g, \tau)$$

with

$$H_{\bar{m}} < 0^2; H_{\bar{b}} > 0; 0 < H_\pi < 1; H_{\bar{g}} > 0; H_\tau < 0$$

We then substitute these equations into 6.71 and 6.72, noting that under this financing rule, $\frac{d\bar{b}}{dt} = 0$.

This gives

$$\dot{\pi} = \alpha \left(F(\bar{m}, \bar{\bar{b}}, \pi, g, \tau) - 1 \right)$$

$$\frac{d\bar{m}}{dt} = \bar{g} + \bar{\bar{b}} \left(H(\bar{m}, \bar{\bar{b}}, \pi, g, \tau) - \pi - n \right) - \tau F(\bar{m}, \bar{\bar{b}}, \pi, g, \tau) - (\pi + n)\bar{m}$$

The linear approximation of this dynamic system at the steady state is

$$\begin{bmatrix} \dot{\pi} \\ \frac{d\bar{m}}{dt} \end{bmatrix} \approx \begin{bmatrix} \alpha F_\pi & \alpha F_{\bar{m}} \\ \bar{\bar{b}}(H_\pi - 1) - \tau F_\pi - \bar{m} & \bar{\bar{b}} H_{\bar{m}} - \tau F_{\bar{m}} - (\pi + n) \end{bmatrix} \begin{bmatrix} \pi - \tilde{\pi} \\ \bar{m} - \widetilde{\bar{m}} \end{bmatrix} + \begin{bmatrix} \alpha F_{\bar{b}} & \alpha F_{\bar{g}} & \alpha F_\tau \\ r - n + \bar{\bar{b}} H_{\bar{b}} - \tau F_{\bar{b}} & 1 + \bar{\bar{b}} H_{\bar{g}} - \tau F_{\bar{g}} & \bar{\bar{b}} H_\tau - \tau F_\tau - \bar{y} \end{bmatrix} \begin{bmatrix} \bar{\bar{b}} - \widetilde{\bar{\bar{b}}} \\ \bar{g} - \widetilde{\bar{g}} \\ \tau - \widetilde{\tau} \end{bmatrix}$$

Both state variables are predetermined. Stability requires two characteristic roots with negative real parts, that is,

$$\alpha F_\pi \left(\bar{\bar{b}} H_{\bar{m}} - \tau F_{\bar{m}} - (\pi + n) \right) - \left(\bar{\bar{b}}(H_\pi - 1) - \tau F_\pi - \bar{m} \right) \alpha F_{\bar{m}} > 0 \quad (6.77a)$$

and

$$\alpha F_\pi + \bar{\bar{b}} H_{\bar{m}} - \tau F_{\bar{m}} - (\pi + n) < 0 \quad (6.78)$$

Neither 6.77a nor 6.78 is automatically satisfied. One destabilising influence is that an increase in the (expected) rate of inflation reduces the real interest rate in the short run in this neo-Keynesian model ($0 < H_\pi < 1$ and $F_\pi > 0$). This boosts demand and, through the Phillips curve, inflation.

6.3.3 Constant Trend Velocity and Endogenous Public Interest-Bearing Debt

Now assume that the authorities issue or retire nominal money in the exact amount required to keep $\bar{m}$, the ratio of money to trend GDP or the reciprocal of the trend income velocity of circulation of money, constant, that is, the growth rate of the nominal money stock, $\mu \equiv \frac{\dot{M}}{M} = \pi + n$. The nominal interest rate and public interest-bearing debt issuance now are endogenous.

The dynamic system now is:

$$\dot{\pi} = \alpha \left(F(\overline{\overline{m}}, \bar{b}, \pi, g, \tau) - 1 \right)$$

$$\frac{d\bar{b}}{dt} = \bar{g} + \bar{b} \left(H(\overline{\overline{m}}, \bar{b}, \pi, g, \tau) - \pi - n \right) - \tau F(\overline{\overline{m}}, \bar{b}, \pi, g, \tau) - (\pi + n)\overline{\overline{m}}$$

The linear approximation of this dynamic system at the steady state is

$$\begin{bmatrix} \dot{\pi} \\ \frac{d\bar{b}}{dt} \end{bmatrix} \approx \begin{bmatrix} \alpha F_{\pi} & \alpha F_{\bar{b}} \\ \bar{b}(H_{\pi} - 1) - \tau F_{\pi} - \overline{\overline{m}} & r - n + \bar{b} H_{\bar{b}} - \tau F_{\bar{b}} \end{bmatrix} \begin{bmatrix} \pi - \tilde{\pi} \\ \bar{b} - \tilde{\bar{b}} \end{bmatrix}$$
$$+ \begin{bmatrix} \alpha F_{\bar{m}} & \alpha F_{\bar{g}} & \alpha F_{\tau} \\ -(\pi + n) + \bar{\bar{b}} H_{\bar{m}} - \tau F_{\bar{m}} & 1 + \bar{\bar{b}} H_{\bar{g}} - \tau F_{\bar{g}} & \bar{\bar{b}} H_{\tau} - \tau F_{\tau} - \bar{y} \end{bmatrix} \begin{bmatrix} \overline{\overline{m}} - \tilde{\overline{\overline{m}}} \\ \bar{g} - \tilde{\bar{g}} \\ \tau - \tilde{\tau} \end{bmatrix}$$

Both state variables again are predetermined. Stability requires two characteristic roots with negative real parts, that is,

$$\alpha F_{\pi} \left(r - n + \bar{b} H_{\bar{b}} - \tau F_{\bar{b}} \right) - \left(\bar{b}(H_{\pi} - 1) - \tau F_{\pi} - \overline{\overline{m}} \right) \alpha F_{\bar{b}} > 0 \qquad (6.79a)$$

and

$$\alpha F_{\pi} + \bar{b} H_{\bar{m}} + r - n + \bar{b} H_{\bar{b}} - \tau F_{\bar{b}} < 0 \qquad (6.80)$$

The system is highly unlikely to be stable with endogenous bond financing. The trace condition (6.80) is unlikely to be satisfied especially if $r > n$ and if $F_{\bar{b}} < 0$ (the shift of the LM curve to the left when $\bar{b}$ increases dominates the shift of the IS curve to the right).

6.3.4 A Fixed Nominal Interest Rate

Finally, consider the case where the authorities peg the nominal interest rate, $i = \bar{\imath} > 0$. Following the earlier analysis of the nominal interest rate peg in the fixed price level model, the composition of the public debt between M and B now becomes a short-run endogenous variable. We solve the IS and LM curves for $\bar{y}$ and $\bar{m} \equiv \frac{M}{PY}$as functions of $\bar{\pounds} \equiv \frac{M+B}{PY}$, π, $\bar{g}$ and τ.

This yields

$$\bar{y} = F(\bar{\pounds}, \pi; i, \bar{g}, \tau)$$

$$\bar{m} = J(\bar{\pounds}\,, \pi; i, \bar{g}, \tau)$$

It follows from 6.66 that

$$F_{\overline{\pounds}} = D^{-1}(c_A - h_A i c_{Y_d}) \gtreqless 0$$

we assume, as before, that

$$F_{\overline{\pounds}} > 0$$

$$J_{\overline{\pounds}} = D^{-1}\left[(1-(1-\tau)c_{Y_d} - \iota_Y)\, h_A + h_y c_A\right] > 0$$

[3]

As before, we assume that

$$0 < J_{\overline{\pounds}} < 1$$

$$F_\pi = -D^{-1}(c_r + \iota_r) > 0$$

$$J_\pi = -D^{-1} h_Y (c_r + \iota_r) > 0$$

[3]Note that because of the linear homogeneity of our decision rules in the scale variables, $\iota_y = 1$.

$$F_i = D^{-1}\left(c_r + \iota_r + \bar{b}c_{Y_d} - ic_{Y_d}h_i\right) \gtreqless 0$$

We assume

$$F_i < 0$$

$$J_i = D^{-1}\left[\left(1-(1-\tau)c_{Y_d} - \iota_Y\right)h_i + h_A(c_r + \iota_r + \bar{b}c_{Y_d})\right] \gtreqless 0$$

We assume

$$J_i < 0$$

$$F_{\bar{g}} = D^{-1} > 0$$

$$J_{\bar{g}} = D^{-1}h_Y > 0$$

$$F_\tau = -D^{-1}\bar{y}c_{Y_d} < 0$$

$$J_\tau = -D^{-1}h_Y\bar{y}c_{Y_d} < 0$$

$$D = 1-(1-\tau)c_{Y_d} - \iota_Y + ic_{Y_d}h_Y > 0$$

This gives the following dynamic system in the two predetermined state variables π and $\bar{\pounds}$:

$$\dot{\pi} = \alpha\left(F(\bar{\pounds},\pi,i,\bar{g},\tau) - 1\right)$$

$$\frac{d\bar{\pounds}}{dt} = \bar{g} + i\left(\bar{\pounds} - J(\bar{\pounds},\pi,i,\bar{g},\tau)\right) - \tau F(\bar{\pounds},\pi,i,\bar{g},\tau) - (\pi+n)\bar{\pounds}$$

A linear approximation at the steady state yields:

$$\begin{bmatrix} \dot{\pi} \\ \frac{d\bar{\pounds}}{dt} \end{bmatrix} \approx \begin{bmatrix} \alpha F_{\pi} & \alpha F_{\bar{\pounds}} \\ -iJ_{\pi} - \tau F_{\pi} - \bar{\pounds} & i(1 - J_{\bar{\pounds}}) - \tau F_{\bar{\pounds}} - (\pi + n) \end{bmatrix} \begin{bmatrix} \pi - \tilde{\pi} \\ \bar{\pounds} - \widetilde{\bar{\pounds}} \end{bmatrix}$$
$$+ \begin{bmatrix} \alpha F_{i} & \alpha F_{\bar{g}} & \alpha F_{\tau} \\ \bar{b} - iJ_{i} - \tau F_{i} & 1 - iJ_{\bar{g}} - \tau F_{\bar{g}} & -iJ_{\tau} - \tau F_{\tau} - \bar{y} \end{bmatrix} \begin{bmatrix} i - \tilde{\imath} \\ \bar{g} - \widetilde{\bar{g}} \\ \tau - \widetilde{\tau} \end{bmatrix}$$

This will be locally stable if and only if

$$\alpha F_{\pi} \left(i(1 - J_{\bar{\pounds}}) - \tau F_{\bar{\pounds}} - (\pi + n)\right) + \left(iJ_{\pi} + \tau F_{\pi} + \bar{\pounds}\right) \alpha F_{\bar{\pounds}} > 0$$

and

$$\alpha F_{\pi} + i(1 - J_{\bar{\pounds}}) - \tau F_{\bar{\pounds}} - (\pi + n) < 0$$

Again, the stability conditions are not necessarily satisfied.

The moral of all this is that, for models which have an accelerationist Phillips curve (or some generalisation of this), exogenous or '*open-loop*' policies are likely to be associated with unstable behaviour. Policy has to 'feed back' appropriately from state variables, such as the inflation rate and or debt stocks, in order to stabilise the system.

Chapter 7

A Simple Forward-Looking Monetary Model

7.1 A discrete Time Monetary Model

A useful reference for the continuous time version of this model is Sargent and Wallace [24]

We consider a simplified version of the flexible price level version of the AD-AS model.

$$\begin{gathered} m(t) - p(t) = ky(t) - \lambda i(t) \\ \lambda > 0 \end{gathered} \tag{7.1}$$

$$y(t) = \bar{y} = 0 \tag{7.2}$$

$$i(t) = r(t) + E_t p(t+1) - p(t) \tag{7.3}$$

$$r(t) = \bar{r} = 0 \tag{7.4}$$

All variables other than interest rates are in natural logarithms.

The price level is assumed to be non-predetermined. Its current value is not inherited from the

past, but can respond to current news about future events. The boundary condition to select a unique solution takes the form of a terminal condition rather than an initial condition.

Equations 7.1 to 7.4 imply the following first-order linear difference equation with constant
coefficients:

$$p(t) = \frac{\lambda}{1+\lambda} E_t p(t+1) + \frac{1}{1+\lambda} m(t) \tag{7.5}$$

Substitute recursively forward for $E_t p(t+1)$, $E_{t+1} p(t+2)$ etc. in 7.5, using the 'Law of Iterated Projections' given in7.6 below:

$$\begin{aligned} E_t E_{t+j} p(t+k) &= E_t p(t+k) \text{ if } j \geq 0 \\ &= E_{t+j} p(t+k) \text{ if } j < 0 \end{aligned} \tag{7.6}$$

Equation 7.6 is a direct implication of a fundamental property of conditional expectations, given in 7.7 below, provided $\Omega_t \subseteq \Omega_{t+j}$ for $j \geq 0$, where Ω_t is the information set conditioning expectations formed at time t.

$$E\left[E\left(y|\, x, z\right)|\, z\right] = E\left(y|\, z\right) \tag{7.7}$$

We get the following expression for the current price level as a function of the current money stock, current expectations of future money stocks and a nuisance term, $B(t)$:

$$p(t) = \frac{1}{1+\lambda} m(t) + \frac{1}{1+\lambda} \sum_{j=1}^{\infty} \left(\frac{\lambda}{1+\lambda}\right)^j E_t m(t+j) + B(t)$$

$$B(t) = \lim_{j \to \infty} \left(\frac{\lambda}{1+\lambda}\right)^j E_t p(t+j)$$

Any stochastic process for B which satisfies the homogeneous equation of 7.5, that is, any process for B satisfying

$$E_t B(t+1) = \left(\frac{1+\lambda}{\lambda}\right) B(t) \tag{7.8}$$

can be part of a rational expectations solution for the price level. Note that, since $\lambda > 0$, the 'bubble term', $B(t)$, will explode in expectations unless, at the initial date, t_0, $B(t_0) = 0$. In what follows we will make this assumption, which rules out rational speculative bubbles. This is the 'terminal'

boundary condition that allows us to find a unique solution for our difference equation. The resulting 'bubble-free' solution is called a *fundamental* solution.

$$p(t) = \frac{1}{1+\lambda} m(t) + \frac{1}{1+\lambda} \sum_{j=1}^{\infty} \left(\frac{\lambda}{1+\lambda} \right)^{j} E_t m(t+j) \tag{7.9}$$

7.1.1 Direct Effects, Announcement Effects and the 'Lucas Critique'

The total effect of an increase in the current money stock, $m(t)$, on the price level, is the sum of the direct effect (holding constant expectations of future variables) and announcement effects (effects working through changes in expectations about future variables):

$$\begin{aligned} \frac{dp(t)}{dm(t)} &= \frac{1}{1+\lambda} \\ &\quad + \frac{1}{1+\lambda} \sum_{j=1}^{\infty} \left(\frac{\lambda}{1+\lambda} \right)^{j} \frac{\partial E_t m(t+j)}{\partial m(t)} \end{aligned}$$

Consider e.g. the following first-order autoregressive process for the (logarithm of) the nominal money stock:

$$m(t+1) = \beta m(t) + \varepsilon(t+1) \tag{7.10}$$

ε is a white noise random disturbance. This implies $E\varepsilon = 0$
From 7.10 it follows that

$$\begin{gathered} E_t m(t+j) = \beta^j m(t) \\ j \geq 1 \end{gathered} \tag{7.11}$$

It follows that, for $j \geq 1$,

$$\frac{\partial E_t m(t+j)}{\partial m(t)} = \beta^j$$

Substituting 7.9 into 7.11 gives

$$p(t) = m(t)\left(\frac{1}{1+\lambda}\right)\sum_{j=0}^{\infty}\left(\frac{\lambda\beta}{1+\lambda}\right)^j$$

Provided $\left|\frac{\lambda\beta}{1+\lambda}\right| < 1$, the price level is therefore given by

$$p(t) = \frac{1}{1+\lambda(1-\beta)}m(t) \tag{7.12}$$

Note that the relationship between the equilibrium price level and the current money stock depends both on the parameter of the private decision rule (λ, the semi-elasticity of the money demand function with respect to the nominal interest rate) and the parameter of the monetary policy rule, β. This is an example of the *Lucas critique* (see e.g. Lucas [23]). When the policy rule changes, the 'reduced form' relationship in the data between private choice variables and policy instruments can change because current changes in policy instruments may affect private expectations about future values of the policy instrument and because with forward-looking private behaviour, current private decisions may depend on current expectations of future policy instrument values.

The total effect of a change in the current money stock on the price level is given by

$$\begin{aligned}\frac{dp(t)}{dm(t)} &= \frac{1}{1+\lambda(1-\beta)}\\ &= \left.\frac{\partial p(t)}{\partial m(t)}\right|_{\{E_t m(t+j);j\geq 1\}}\\ &\quad + \sum_{j=1}^{\infty}\frac{\partial p(t)}{\partial E_t m(t+j)}\frac{\partial E_t m(t+j)}{\partial m(t)}\\ &= \frac{1}{1+\lambda}\\ &\quad + \sum_{j=1}^{\infty}\left(\frac{\lambda}{1+\lambda}\right)^j\beta^j\end{aligned}$$

Consider some special cases:

(a) $\beta = 1$

The logarithm of the nominal money stock follows a martingale (random walk). All shocks to the money stock are perceived to be permanent: $E_t\varepsilon(t+j) = \varepsilon(t)$ for all $j \geq 1$.

In this case

$$p(t) = m(t)$$

(b) $\beta = 0$

The logarithm of the nominal money stock is white noise. All shocks to the money stock are perceived as temporary: $E_t\varepsilon(t+j) = 0$ for all $j \geq 1$.

In this case,

$$p(t) = \frac{1}{1+\lambda}m(t)$$

There are no announcement effects associated with changes in the current money stock. The direct effect is the same as the total effect. An increase in the current money stock leads to a less than proportional increase in the price level. Since future money stocks are unchanged, the future price level is expected to be lower than the current price level. This negative expected rate of inflation increases the demand for real money balances, so the current price level increases by less (proportionally) than the current money stock.

(c) $\beta > 1$. Note that monetary growth cannot be too explosive, since we require $\left|\frac{\lambda\beta}{1+\lambda}\right| < 1$. In this case $\frac{dp(t)}{dm(t)} > 1$.

(d) $\beta < 0$. In this case a positive shock to the current money stock leads to rational expectations of future reductions in the money stock. We have $0 < \frac{dp(t)}{dm(t)} < \frac{1}{1+\lambda}$.

Exercise 7.1.1 *Can β be sufficiently negative to cause the current price level to fall when the current money stock increases? Explain your answer.*

For a constant value of the money stock, say $m = \bar{m}$, and dropping the expectation operator, we can draw the Graph of the pricing function in $p(t+1) - p(t)$ space, as in Figure 7.1

Figure 7.1 here

For a constant value of the exogenous money stock, the only value of p that does not place the economy on an explosive trajectory is the intersection of the price graph and the 45^0 line.

7.2 A Continuous-Time Monetary Model

The continuous time version of the model analysed in the previous section is given below

$$m(t) - p(t) = ky(t) - \lambda^{-1}i(t)$$
$$\lambda > 0$$

$$y(t) = \bar{y} = 0$$

$$i(t) = r(t) + \dot{p}(t)$$

$$r(t) = \bar{r} = 0$$

This implies the following differential equation

$$\dot{p} = \lambda p - \lambda m \tag{7.13}$$

Since I want to consider policy changes that lead to changes in steady state inflation, it is convenient to choose as the state variable the real stock of money, $\ell \equiv m - p$. The growth rate of the nominal money stock is denoted $\mu \equiv \dot{m}$.

We can therefore rewrite 7.13 as

$$\dot{\ell} = \lambda \ell + \mu \tag{7.14}$$

In steady state,

$$\ell = -\lambda^{-1} \mu \tag{7.15}$$

In steady state, a higher growth rate of the nominal money stock is associated with a higher rate of inflation, a higher nominal interest rate and a lower stock of real money balances.

Mathnote

Consider the following first-order linear differential equation . x is a non-predetermined or forward-looking state variable, z is an exogenous or forcing variable.

$$\dot{x}(t) = a(t)x + b(t)z(t)$$

The forward-looking solution is

$$x(t) = b\int_t^\infty e^{-\int_t^s a(u)du} z(s)ds + ke^{\int_{t_o}^t a(u)du}$$

where k is an arbitrary constant to be determined by a boundary condition.

If the coefficients are constant, this simplifies to

$$x(t) = b\int_t^\infty e^{-a(s-t)} z(s)ds + ke^{at}$$

The boundary condition is

If there exists a continuously convergent solution for x,choose it.

If $a > 0$, this boundary condition implies $k = 0$.
This then yields the unique *fundamental* forward-looking solution:

$$x(t) = b\int_t^\infty e^{-a(s-t)} z(s)ds$$

The effects of changes in the growth rate of the nominal money stock are conveniently analysed in a diagram with ℓ on the horizontal axis and $\dot{\ell}$ on the vertical axis. In models with forward-looking rational expectations, one has to be very precise about the exact specification of the shock hitting the system.

The first issue to be resolved is how it is logically possible to analyse *shocks,* that is, unexpected events, in a 'perfect foresight' model (a rational expectations model without explicit uncertainty). The resolution of this paradox involves two steps

(1) The agent forming the expectations solves correctly for the model-consistent expectations of the endogenous variables, conditional on his expectations of the exogenous variables.

(2) The expectations of exogenous variables are held with complete subjective certainty. In Bayesian terms, if the optimal forecast is the mean of his posterior distribution (not to be confused the distribution of his posterior), all probability is concentrated on a single 'mass point'. These subjectively held expectations of exogenous variables can, however, be wrong. When they are revised, they are, however, again held with complete subjective certainty.

A shock is therefore effectively a zero probability event that nevertheless happens (don't ask, it makes some sense as a limiting argument).

When news about the exogenous variables accrues at t_0 , say (the announcement date), we have to specify (a) the complete current and future path of the exogenous variables expected just before t_0, at $t_0^{(-)}$, say and (b) the complete current and future path of the exogenous variables expected from t_0 on. We can then solve the model until the occurrence of the next shock.

Some examples follow. We always start from a steady state and the expectation at $t_0^{(-)}$ of future monetary growth is for a constant monetary growth rate, $\mu = \bar{\mu}$. Our boundary condition is that we should pick a continuously convergent solution, if one exists. That means that there can only be a discrete change in the price level, p, and, given m, a discrete change in ℓ, at the 'announcement date', t_0. At the future 'implementation date(s)' when the unexpected changes in μ announced at t_0 actually come into effect, there can be no discrete jumps in ℓ. In the diagram, this means that there can only be one horizontal discrete change the $\dot{\ell}, \ell$ space, at $t = t_0$. There can be subsequent discrete (vertical) jumps in $\dot{\ell}$.

The economic rationale for restricting discrete jumps in the price *level* to those instants at which news arrives is a speculative efficiency argument. The restriction rules out anticipated future discrete jumps in the price level. If such a future discrete jump in p were anticipated (consider for concreteness a discrete fall in p at some future date t_1, .agents would have a huge incentive to put in enormous sales orders for the good the instant before t_1, confidently expecting to be able to buy the goods again the next instant at the lower price. The instantaneous rate of return to this transaction would be infinite. Every one would rush in with sell orders the instant before t_1 and the discrete fall in the price level would in fact occur the instant before t_1. By backward induction, it follows that the discrete price level fall would occur the instant that news justifying it arrives.

For all three univariate examples, the mode of analysis is the same. Once the last announced change in μ has been implemented, and μ is expected to be constant forever after, the system has to be at the steady state equilibrium corresponding to that final constant value of μ. From that final steady state, we now work backwards to the single initial horizontal jump in ℓ that will enable the system to reach the final steady state while satisfying the appropriate equation of motion at each instant of the transition.

(I) The unexpected (and credible) announcement at $t = t_0$ of an immediate, permanent reduction in the growth rate of money to $\bar{\bar{\mu}} < \bar{\mu}$, .that is,

$$\begin{aligned}
&E_{t_0^{(-)}}\mu(t) = \bar{\mu}, \;\; t \geq t_0 \\
&E_{t_0}\mu(t) = \overline{\overline{\mu}}, \;\; t \geq t_0
\end{aligned}$$

Figure 7.2 here

The transition to the new steady state is instantaneous.. The price level falls discretely at t_0. When the growth rate of money is expected to be constant forever, the system is always in steady state. Therefore, from 7.15 it follows that, before t_0

$$p(t) = m(t) + \lambda^{-1}\bar{\mu}$$

From t_0 on,

$$p(t) = m(t) + \lambda^{-1}\overline{\overline{\mu}}$$

(II) The unexpected (credible) announcement at $t = t_0$ of a future, permanent reduction in the growth rate of money to $\overline{\overline{\mu}} < \bar{\mu}$, at $t_1 > t_0$, that is,

$$\begin{aligned}
&E_{t_0^{(-)}}\mu(t) = \bar{\mu}, \;\; t \geq t_0 \\
&E_{t_0}\mu(t) = \bar{\mu}, \;\; t_0 \leq t < t_1 \\
&E_{t_0}\mu(t) = \overline{\overline{\mu}}, \;\; t \geq t_1
\end{aligned}$$

Figure 7.3 here

(III) The unexpected (credible) announcement at $t = t_0$ of a future temporary reduction in the growth rate of money, starting at $t_1 > t_0$ and reversed at $t_2 > t_1$, that is,

$$\begin{aligned}
&E_{t_0^{(-)}}\mu(t) = \bar{\mu}, \;\; t \geq t_0 \\
&E_{t_0}\mu(t) = \bar{\mu}, \;\; t_0 \leq t < t_1 \\
&E_{t_0}\mu(t) = \overline{\overline{\mu}}, \;\; t_1 \leq t < t_2 \\
&E_{t_0}\mu(t) = \bar{\mu}, \;\; t \geq t_2
\end{aligned}$$

Figure 7.4 here

Note that, between t_0 and t_1, p is falling at an increasing rate, since $\dot{p} = -\lambda\ell$ and ℓ is rising at an increasing rate.

7.3 Unpleasant Monetarist Arithmetic

Useful references for this Section are Sargent and Wallace [25] and Buiter [10].

The demand for money is given by

$$\begin{aligned} \frac{M}{PY} &= \alpha_0 - \alpha_1 i \\ \alpha_0 &> 0;\ \alpha_1 \geq 0; M \geq 0 \end{aligned} \tag{7.16}$$

[1]

$$i = r + \pi$$

The real interest rate is exogenous and constant

$$r = \bar{r}$$

Real GDP grows at the exogenous rate n, that is,

$$\dot{Y}/Y = n$$

Until further notice, we assume that government debt is index-linked. β is the stock of real government debt. The government budget identity is:

$$\frac{\dot{M}}{PY} + \frac{\dot{\beta}}{Y} \equiv g + r\frac{\beta}{Y} - \tau \tag{7.17}$$

where g is government spending as a fraction of GDP and τ is net taxes as a fraction of GDP. We also define:

$$m \equiv \frac{M}{PY}$$

[1] The original Sargent and Wallace model is an overlapping generations (OLG) model. The population is divided into two classes. The rich hold real capital and government interest-bearing debt as stores of value, as long as the nominal interest rate, $i = r + \pi$, is positive. The poor can only hold money. The demand for money therefore depends on the (expected) rate of inflation rather than on the nominal interest rate. The specification adopted here does not result in qualitatively different behaviour.

$$b \equiv \frac{\beta}{Y}$$

This permits us to rewrite the government budget identity as

$$\begin{aligned} \dot{m} + \dot{b} &\equiv g - \tau + ib - (\pi + n)(m + b) \\ &\equiv g - \tau + (r - n)b - (\pi + n)m \end{aligned} \tag{7.18}$$

7.3.1 Endogenous Borrowing

Under this regime the authorities fix the growth rate of the nominal money stock, $\mu \equiv \frac{\dot{M}}{M}$. With the government primary deficit (as a fraction of GDP), δ, exogenously given,

$$\delta \equiv g - \tau$$

the nominal interest rate and the amount of borrowing are endogenously determined.

Noting that

$$\dot{m} \equiv (\mu - \pi - n)m \tag{7.19}$$

the government budget identity becomes

$$\dot{b} \equiv \delta + (r - n)b - \mu m \tag{7.20}$$

From the monetary equilibrium condition,

$$\pi = -r + \alpha_1^{-1}\alpha_0 - \alpha_1^{-1}m \tag{7.21}$$

$$\dot{m} = -\alpha_2 m + \alpha_1^{-1}m^2$$

$$\alpha_2 = \alpha_1^{-1}\left(\alpha_0 - \alpha_1(r - n + \mu)\right) \tag{7.22a}$$

Figure 7.5 here

There are two stationary equilibria. The first, rather sad, equilibrium has $m = 0$. It is, however, a locally stable equilibrium. The economy is demonetised. For an economically sensible solution, we require $m > 0$. This requires us to restrict permissible parameter values to satisfy $\alpha_2 \equiv \alpha_1^{-1}(\alpha_0 - \alpha_1(r - n + \mu)) > 0$. The second steady state is $m = \alpha_1\alpha_2 > 0$. It is unstable.

Using the transformation $v = m^{-1}$, we can rewrite 7.24 as follows:

$$\dot{v} = \alpha_2 v - \alpha_1^{-1}$$

Like P and m, the income velocity of circulation of money, v, is a non-predetermined state variable. We choose the forward-looking solution

$$v(t) = \alpha_1^{-1}\int_t^{\infty} e^{-\int_t^s \alpha_2(v)dv} ds + k \lim_{s\to\infty} e^{-\int_t^s \alpha_2(v)dv} ds \tag{7.23}$$

Choosing the unique continuously convergent solution for v, we set the arbitrary constant $k = 0$. The solution for velocity becomes

$$v(t) = \alpha_1^{-1}\int_t^{\infty} e^{-\int_t^s \alpha_2(v)dv} ds$$

This implies that the solution for the money-GDP ratio is

$$m(t) = \alpha_1\left(\int_t^{\infty} e^{-\int_t^s \alpha_2(v)dv} ds\right)^{-1} \tag{7.24}$$

When α_2 is constant, this simplifies to

$$m(t) = \alpha_1\alpha_2 = \alpha_0 - \alpha_1(r - n + \mu) \tag{7.25}$$

Substituting 7.25 into 7.19 yields:

$$\dot{b} = (r - n)b + \delta - \mu(\alpha_0 - \alpha_1(r - n + \mu)) \tag{7.26}$$

It follows that, if $r > n$, the debt dynamics will be explosive. This is shown in Figure 7.6.

Figure 7.6 here

The economy is always on the $m = \alpha_1\alpha_2$ schedule. The steady state, Ω, is unstable.

7.3.2 Endogenous Money Growth

We now consider the case where the authorities issue or retire interest-bearing debt in such as way as to keep the interest-bearing debt-GDP ratio constant, that is

$$\dot{\beta} = n\beta$$

or

$$b = \bar{b}$$

Monetary issuance and the nominal interest rate are therefore endogenous.

Under this rule,

$$\dot{m} \equiv \delta + (r-n)\bar{b} - (\pi + n)m$$

or, using the money demand function to eliminate π,

$$\dot{m} \equiv \delta + (r-n)\bar{b} - \left(\frac{\alpha_0}{\alpha_1} - r + n\right)m + \frac{1}{\alpha_1}m^2$$

The two steady states of this model are the roots of the above quadratic:

$$m = \frac{1}{2}\left(\alpha_0 - \alpha_1(r-n) \pm \sqrt{[\alpha_0 - \alpha_1(r-n)]^2 - 4\alpha_1[\delta + (r-n)\bar{b}]}\right)$$

There will be two steady states if $\delta + (r-n)\bar{b}$, the 'inflation and real growth-corrected deficit' (as a fraction of GDP) is not too large: $[\alpha_0 - \alpha_1(r-n)]^2 - 4\alpha_1[\delta + (r-n)\bar{b}] > 0$. The low m, high inflation, steady state is locally

stable. The high m, low inflation steady state is unstable. We cannot rule out the high-inflation, low m equilibrium on the grounds that the steady state money stock has to be positive, if $\alpha_0 - \alpha_1(r-n) > 0$, since in that case both steady states have positive steady state money stocks. An increase in the primary deficit-GDP ratio, δ, shifts the parabola up vertically by the amount of the increase in δ. The high m, low inflation steady state moves from Ω_0 to Ω_1, so the steady-state inflation rate rises. The low m, high inflation steady state, however, moves from Ω_0' to Ω_1'. The steady-state inflation rate falls.

Figure 7.7 here

What happens is that the linear money demand function, with its constant semi-elasticity of money demand with respect to the nominal interest rate, has a long-run seigniorage Laffer curve: the inflation tax revenues πm first rise with the rate of inflation, for low values of the inflation rate, but fall with the rate of inflation for sufficiently high rates of inflation. In other words, the (absolute value of the) elasticity of long-run money demand with respect to the rate of inflation is below unity for low rates of inflation but above unity for high rates of inflation. The low m, high inflation steady state is on the 'wrong' or 'slippery' side of the long-run seigniorage Laffer curve. Following Sargent and Wallace, I shall focus on the unstable steady state, which is on the 'right' side of the long-run seigniorage Laffer curve.

Using the money demand function, we can use an equivalent representation of this model using inflation as the state variable. This yields

$$\dot{\pi} = -\pi^2 + (\frac{\alpha_0}{\alpha_1} - r - n)\pi - \frac{1}{\alpha_1}[\delta + (r-n)\bar{b} + n(\alpha_1 r - \alpha_0)]$$

The two steady-state inflation rates are

$$\pi = \frac{1}{2}\left(\frac{\alpha_0}{\alpha_1} - r - n \pm \sqrt{\left(\frac{\alpha_0}{\alpha_1} - r - n\right)^2 - \frac{4}{\alpha_1}[\delta + (r-n)\bar{b} + n(\alpha_1 r - \alpha_0)]}\right) \tag{7.27}$$

The low-inflation steady state (which is the unstable one) again has the property that a higher value of $\delta + (r-n)\bar{b}$ is associated with a higher rate of inflation.

Sargent and Wallace then conduct the following thought experiment. We start at $t = 0$. Assume that the authorities follow, for a period of duration

T, a policy of fixing the nominal growth rate of money and determining the growth rate of the public debt residually. After an interval of duration T the authorities stabilise the debt-GDP ratio at whatever level prevails at time $t = T$. After period T, monetary growth therefore becomes endogenous.

From equation 7.26 we know that

$$b(T) = b(0)e^{(r-n)T} + \left(\delta - [\alpha_0 - \alpha_1(r-n)]\mu + \alpha_1\mu^2\right) \int_0^T e^{(r-n)(T-s)} ds$$

or

$$b(T) = b(0)e^{(r-n)T} + \left(\delta - [\alpha_0 - \alpha_1(r-n)]\mu + \alpha_1\mu^2\right) K$$

$$K = \frac{e^{(r-n)T}}{r-n}\left(1 - e^{-(r-n)T}\right) > 0 \text{ if } r > n \tag{7.28}$$

This implies that

$$\frac{\partial b(T)}{\partial \mu} = -K\left[\alpha_0 - \alpha_1(r-n) - 2\alpha_1\mu\right]$$

For small values of α_1 and for small values of μ this will be negative. In particular, if $\alpha_1 = 0$ (the demand for money is independent of the nominal interest rate or the velocity of money is constant), this expression will be zero. Lower monetary growth during the interval $(0, T)$ will therefore result in a higher debt-GDP ratio at T (and beyond). From 7.27 it then follows that, if $r > n$, inflation and money growth will be higher than it would otherwise have been. Thus without a 'fundamental' fiscal correction (a reduction in δ) a reduction in monetary growth leads to a long-run increase in the rate of inflation. The additional debt issued during the period of lower money growth means that a higher growth rate of money after time T is required to service the higher debt interest bill.

What happens to inflation in the short run, that is, before $t = T$, when the lower growth rate of money is in effect?

If $\alpha_1 = 0$, the answer is obvious. In that case, $\pi = \mu - n$, and in the short run, inflation comes down when money growth is temporarily reduced without any fundamental fiscal correction.

When $\alpha_1 > 0$, it is possible that the inflation rate rises even during the period that money growth is lower. We show this as follows. From the money

demand function, we know that the rate of inflation is a decreasing function of the stock of real money balances:

$$\pi(t) = r + \alpha_1^{-1}\alpha_0 - \alpha_1^{-1}m(t)$$

From 7.24 and 7.22a, we know that

$$m(t) = \alpha_1 \left(\int_t^{\infty} e^{-\int_t^s \alpha_1^{-1}(\alpha_0 - \alpha_1[r-n+\mu(v)])dv} ds \right)^{-1}$$

so

$$\pi(t) = r + \alpha_1^{-1}\alpha_0 - \left(\int_t^{\infty} e^{-\int_t^s \alpha_1^{-1}(\alpha_0 - \alpha_1[r-n+\mu(v)])dv} ds \right)^{-1} \qquad (7.29)$$

This means that current inflation is an increasing function of all current and future growth rates of the nominal money stock. In our example, the growth rate of nominal money is reduced for $0 \leq t < T$, but, if $r > n$ and the other conditions (being on the right side of the long-run seigniorage Laffer curve etc.) are satisfied, the growth rate of nominal money is higher for $t \geq T$. Equation 7.29 can be rewritten, for $t < T$ as

$$\pi(t) = r + \alpha_1^{-1}\alpha_0 - (\Psi_1 + \Psi_2)^{-1} \qquad (7.30)$$

$$\Psi_1 = \int_t^{T} e^{-\int_t^s \alpha_1^{-1}(\alpha_0 - \alpha_1[r-n+\mu(v)])dv} ds$$

$$\Psi_2 = \int_T^{\infty} e^{-\int_t^s \alpha_1^{-1}(\alpha_0 - \alpha_1[r-n+\mu(v)])dv} ds$$

It is possible, though not necessary for all realistic parameter configurations, that Ψ_1 falls by less when the early growth rate of money is reduced than Ψ_2 increases as a result of the later increase in monetary growth.

7.3.3 A Nominal Interest Rate Rule

The analysis of Sargent and Wallace's unpleasant monetarist arithmetic is simplified considerably if we assume that the authorities use the nominal interest rate as the instrument. Let $\pounds \equiv m + b$ denote the ration of total public financial liabilities, monetary and non-monetary to GDP. Again, with a nominal interest rate peg the composition of $\pounds$ between money and bonds is endogenous. The government budget identity can be rewritten in terms of the dynamics of a single state variable, $\pounds$.

$$\begin{aligned} \dot{\pounds} &\equiv (r-n)\pounds + \delta - im \\ &\equiv (r-n)\pounds + \delta - \alpha_0 i + \alpha_1 i^2 \end{aligned}$$

We again assume $r > n$.

With i policy-determined and r exogenous, inflation is not too hard to determine:

$$\pi(t) = i(t) - r$$

Assume that the authorities fix i at $i(s) = \bar{i}$ for an interval of time of duration T, starting at $t = 0$. They then stabilize $\pounds(t)$ for all $t \geq T$, at the level achieved at $t = T$.

This means that, for $t \geq T$, the nominal interest rate becomes endogenous and has to satisfy

$$0 = (r-n)\pounds(T) + \delta - \alpha_0 i + \alpha_1 i^2$$

That is, for $t \geq T$

$$i(t) = \frac{\alpha_0 \pm \sqrt{\alpha_0^2 - 4\alpha_1(r-n)\pounds(T)}}{2\alpha_1}$$

We only consider the case where the roots are real, and assume that the authorities choose the lower of the two interest rate solutions, that is,

$$i(t) = \bar{\bar{i}} = \frac{\alpha_0 - \sqrt{\alpha_0^2 - 4\alpha_1(r-n)\pounds(T)}}{2\alpha_1} \quad \text{for } t \geq T$$

It then follows that $\frac{\partial \bar{\bar{i}}}{\partial \pounds(T)} > 0$. A higher value of the public debt at time T implies a higher subsequent nominal interest rate and inflation rate.

A key issue is whether $\pounds$ is a predetermined or a non-predetermined state variable.

We know that, with index-linked interest-bearing debt,

$$\pounds \equiv m + b = \frac{M}{PY} + \frac{\beta}{Y}$$

If, however, government interest-bearing debt were nominally denominated, we would have

$$\pounds \equiv m + b = \frac{M}{PY} + \frac{B}{PY}$$

where B is the stock of nominal public interest-bearing debt.

Consider first the case where the government's interest-bearing debt is one-period (strictly speaking instantaneous or zero-maturity) index-linked debt, that is, denominated in units of real output rather than units of money. The real value of the interest-bearing debt, β, is then given, that is, predetermined at each point in time, and so is $b \equiv \beta/Y$.

It follows that $\pounds = b + m = b + \alpha_0 - \alpha_1 i$ is also predetermined.

The solution for $\pounds(t)$, for all $t \geq 0$ is given by

$$\pounds(t) \equiv \pounds(0)e^{(r-n)t} + \int_0^t e^{(r-n)(t-s)}[\delta - \alpha_0 i(s) + \alpha_1 i^2(s)]ds$$

where $\pounds(0) = b(0) + \alpha_0 - \alpha_1 i(0)$ is determined by the initial condition for $b(0)$ and the nominal interest rate at $t = 0$.

Specifically,

$$\begin{aligned}
\pounds(T) &\equiv \pounds(0)e^{(r-n)T} + \int_0^T e^{(r-n)(T-s)}[\delta - \alpha_0\bar{i} + \alpha_1\bar{i}^2]ds \\
&\equiv \pounds(0)e^{(r-n)T} + [\delta - \alpha_0\bar{i} + \alpha_1\bar{i}^2]K
\end{aligned}$$

where $K > 0$ is defined in 12.32.

$$\frac{\partial \pounds(T)}{\partial \bar{i}} = -K(\alpha_0 - 2\alpha_1\bar{i})$$

Note that $\alpha_0 - 2\alpha_1\bar{i} > 0$ if and only if $\frac{\partial(im)}{\partial i} > 0$. This is the case if and only if the elasticity of money demand with respect to the nominal interest rate is less than 1 in absolute value, that is if and only if $\frac{i\partial m}{m\partial i} > -1$. This

means that the economy is on the sensible side of the seigniorage Laffer curve. We assume this to be the case in what follows. Therefore, $\frac{\partial \pounds(T)}{\partial i} < 0$. A lower initial nominal interest rate (and the associated lower initial rate of inflation) leads to a larger stock of real government liabilities at $t = T$. Without a fundamental fiscal correction, the nominal interest rate, and the rate of inflation, will have to be higher for $t \geq T$.

Note that while the rate of inflation and the real stock of money balances are determinate under the nominal interest rate rule, the nominal money stock and the general price level are not, without further assumptions. The money demand function gives us

$$\frac{M(t)}{P(t)Y(t)} = \alpha_0 - \alpha_1 i(t)$$

With $i(t)$ exogenous, $Y(t) = Y(0)e^{nt}$ and $Y(0)$ given, the model only determines $\frac{M(t)}{P(t)}$.

There are two ways to resolve this. The first is to argue that the initial nominal money stock, $M(0)$ is given by history. The entire current and future nominal money stock and price level paths are then determinate. The other resolution is the assumption that the government not only pegs the current and future values of the nominal interest rate, but also the value of the nominal money stock at some point in time. This could either be the initial nominal money stock or some future value of the nominal money stock. The nominal money stock at all other current and future dates remains endogenously determined.

7.3.4 A Fiscal Theory of the Price Level?

Now consider the case where the government's interest-bearing debt is nominally denominated. The stock of nominal interest-bearing debt at time t is $B(t)$. The initial value of the government's nominal liabilities as a share of GDP, is

$$\pounds(0) \equiv \frac{M(0) + B(0)}{P(0)Y(0)} = \frac{B(0)}{P(0)Y(0)} + \alpha_0 - \alpha_1 i(0) \tag{7.31}$$

Even with $Y(0)$, $i(0)$ and $B(0)$ given, $\pounds(0)$ is not predetermined, as long as $B(0) \neq 0$, because the initial price level, $P(0)$ has not been determined.

We can solve the government's budget identity forward in time. This yields:

$$£(t) \equiv \int_t^\infty e^{-(r-n)(s-t)}[\alpha_0 i(s) - \alpha_1 i(s)^2 - \delta]ds + ke^{(r-n)t}$$

where k is an arbitrary constant. We impose the 'no Ponzi finance' boundary condition on the government's fiscal-financial-monetary programme, that is, $k = 0$. This implies

$$£(t) \equiv \int_t^\infty e^{-(r-n)(s-t)}[\alpha_0 i(s) - \alpha_1 i(s)^2 - \delta]ds \tag{7.32}$$

Equation 7.32 is sometimes called the government's solvency constraint or intertemporal budget constraint. It says that the value of the initial stock of public sector financial liabilities, $£(t)$, must equal the present discounted value of current and future primary surpluses, $-\delta(s)$, $s \geq t$, plus the present discounted value of current and future 'central bank profits', $i(s)m(s)$, $s \geq t$.

For $t \leq T$ we then have

$$\begin{aligned} £(t) \quad &\equiv \quad \int_t^T e^{-(r-n)(s-t)}[\alpha_0 \bar{i} - \alpha_1 \bar{i}^2 - \delta]ds \\ &\quad + \int_T^\infty e^{-(r-n)(s-t)}[\alpha_0 \bar{\bar{i}} - \alpha_1 \bar{\bar{i}}^2 - \delta]ds \end{aligned}$$

At the initial date, $t = 0$, we can write the government's solvency constraint as follows

$$\frac{B(0)}{P(0)Y(0)} + \alpha_0 - \alpha_1 i(0) \equiv \int_0^\infty e^{-(r-n)(s-t)}[\alpha_0 i(s) - \alpha_1 i(s)^2 - \delta]ds \tag{7.33}$$

Woodford [**?**] and others treat the government's solvency constraint as an additional equilibrium condition. Provided

$$sgn\{B(0)\} = sgn\{\int_0^\infty e^{-(r-n)(s-t)}[i(s)m(s) - \delta]ds - m(0)\} \tag{7.34}$$

or, in this particular case, provided

$$sgn\{B(0)\} = sgn\{\int_0^\infty e^{-(r-n)(s-t)}[\alpha_0 i(s) - \alpha_1 i(s)^2 - \delta]ds - \alpha_0 - \alpha_1 i(0)\} \tag{7.35}$$

equation 7.33 permits us to solve for the initial price level (given $B(0)$). That initial price level takes on the value required to ensure that the exogenously fixed sequences of primary surpluses and central bank profits, minus the initial value of the real money stock, equal the real value of the government's nominally denominated interest-bearing public debt.

It should be clear why 7.34 or 7.35 is necessary for this to work.

$$\frac{B(0)}{P(0)Y(0)} \equiv \int_0^\infty e^{-(r-n)(s-t)}[\alpha_0 i(s) - \alpha_1 i(s)^2 - \delta]ds - [\alpha_0 - \alpha_1 i(0)] \quad (7.36)$$

Consider 7.36. Unless 7.34 or 7.35 holds, the initial price level would have to be negative, which would pose a problem in most approaches to macroeconomics. Note also that this 'fiscal theory of the price level' could not work if the government's interest-bearing debt were index-linked or real debt.

$$b(0) \equiv \int_0^\infty e^{-(r-n)(s-t)}[\alpha_0 i(s) - \alpha_1 i(s)^2 - \delta]ds - [\alpha_0 - \alpha_1 i(0)] \quad (7.37)$$

Equation 7.37 with $b(0)$ given, as it would be if the bonds were index-linked cannot determine the initial price level. Indeed, with $b(0)$ given, with the sequence of real primary surpluses exogenously given and with the sequence of and current and future nominal interest rates given, Equation 7.37 cannot hold in general. The system is over-determined. This points to the essential flaw in the 'fiscal theory of the price level'. The government's solvency constraint is an *identity*. It has to hold not only for equilibrium sequences of prices and other endogenous variables, but for all sequences of prices and endogenous equilibrium variables. This is true for any budget constraint, intertemporal or other. What this means is that governments cannot arbitrarily fix their sequences of real spending, real taxes and nominal interest rates (or real seigniorage), and hope that the initial price level will be kind enough to price their outstanding liabilities in such a way as to ensure that their intertemporal budget constraint is satisfied. Some element(s) in the sequences of spending, taxes and nominal interest rates (or seigniorage) has to adjust endogenously to ensure that the government's intertemporal budget constraint holds as an identity, that is, is always satisfied.

If we were to follow the Woodford route, and if condition 7.34 or 7.35 is satisfied, there is no unpleasant monetarist arithmetic. The government's intertemporal budget constraint at the initial date can be written as

$$\begin{aligned}\frac{B(0)}{P(0)Y(0)} - [\alpha_0 - \alpha_1 i(0)] &\equiv \int_0^T e^{-(r-n)(s-t)}[\alpha_0 \bar{i} - \alpha_1 \bar{i}^2 - \delta]ds \\ &+ \int_T^\infty e^{-(r-n)(s-t)}[\alpha_0 \bar{\bar{i}} - \alpha_1 \bar{\bar{i}}^2 - \delta]ds\end{aligned}$$

With $B(0)$ given and of the right sign, the initial price level can make the real value of the government's nominal interest-bearing debt equal to whatever is exogenously determined on the RHS of the budget constraint. The government can reduce the nominal interest rate (and with it the inflation rate) for $t < T$ and for $t \geq T$. If we are on the sensible side of the seigniorage Laffer curve, such permanent reductions in the nominal interest rate and in the rate of inflation would reduce the present discounted value of current and future central bank profits (of real seigniorage). Assuming $B(0) > 0$, we would get a correspondingly higher initial price level to maintain equality between the initial real value of the public debt and the present discounted value of future primary surpluses and central bank profits (or real seigniorage).

My interpretation of this is that this is of no economic interest. The overdeterminacy normally associated with an over-determined fiscal-financial-monetary programme (e.g. fixed sequences of δ and of im) is avoided because of the familiar nominal indeterminacy associated with exogenous nominal interest rate rules in models with a flexible price level. The over-determined budgetary rule (fixed sequences of δ and of im), however, is not an acceptable ingredient in a well-posed macroeconomic model. The modeler can only consider fiscal-financial-monetary programmes for which the government's intertemporal budget constraint holds as an identity. The fact that a misspecified (overdetermined) fiscal-financial-monetary programme may, under certain arbitrary restrictions, 'resolve' the indeterminacy normally associated with fixed nominal interest rate rules in flexible price models, is neither here nor there. We still have an ill-posed macroeconomic model. The correct interpretation of what happens when we conduct the unpleasant monetarist arithmetic experiment (lower the nominal interest rate for $0 \leq t < T$) and set the nominal interest rate for $t \geq T$ at the level required to stabilise the ratio of total financial government liabilities to GDP at its time T, level, when there is nominal non-interest-bearing debt outstanding at $t = 0$ is that both the initial price level and the nominal interest rate for $t \geq T$ are indeterminate.

Note that exactly the same 'fiscal theory of the price level' issues arise in the original Sargent and Wallace exercise, when the growth rate of the

nominal money stock is reduced for some interval. If the public interest-bearing debt is nominally denominated, the initial value of the price level could be fixed from the government's intertemporal budget constraint, provided a condition analogous to **??** or **??** is satisfied, even if the growth rates of the nominal money stock were exogenously fixed for all $t \geq 0$, as well as the sequence of real primary surpluses. Again, such an overdetermined fiscal-financial-monetary programme does not belong in a well-posed macroeconomic model.

Chapter 8

Forward-Looking Financial Markets

8.1 The Term Structure of Interest Rates

Useful references are Turnovsky and Miller [28] and Fisher and Turnovsky [14]. Consider an investor choosing between two alternative financial investments. The first is a one-period nominal risk-free bond, yielding a nominal one-period rate of return i. The gross return on 1£ invested in the short nominal bond for one period is therefore $1 + i$. The second is a consol or perpetuity. This is a bond promising a constant nominal coupon payment each period of $\Gamma > 0$. There is no redemption date, the maturity of the bond is infinite. The price of the consol in period t is $P_C(t)$. The total return (the coupon and the resale value) on 1£ invested in the consol is therefore $\frac{\Gamma+P_C(t+1)}{P_C(t)}$. Risk-neutral investors (who also ignore Jensen's inequality) will only hold both financial claims if their expected rates of return are equalised, that is,

$$1 + i(t) = \frac{\Gamma + E_t P_C(t+1)}{P_C(t)} \tag{8.1}$$

In continuous time, the corresponding equilibrium condition is

$$i(t) = \frac{\Gamma + E_t \dot{P}_C(t)}{P_C(t)}$$

or

$$E_t \dot{P}_C(t) = i(t) P_C(t) - \Gamma \tag{8.2}$$

If future interest rates are known, or if expectations are held with complete subjective certainty, 8.2 implies (dropping the expectation operator),

$$P_C(t) = \Gamma \int_t^{\infty} e^{-\int_t^s i(u)du} ds + k e^{\int_{t_0}^t i(u)du}$$

where k is an arbitrary constant.

The long bond price is a forward-looking asset price. We impose the 'no-bubble' boundary condition $k = 0$. We define the long nominal interest rate, i_L as the coupon yield on the consol, that is,

$$i_L \equiv \frac{\Gamma}{P_C}$$

For simplicity, we can choose units such that $\Gamma = 1$. which implies the following relation between the long nominal interest rate and the short nominal interest rates:

$$i_L(t) = \left(\int_t^{\infty} e^{-\int_t^s i(u)du} ds \right)^{-1} \tag{8.3}$$

Thus the current nominal long rate is a forward-looking moving average of expected future short nominal interest rates. The (expected) rate of return equalisation condition can also be written as

$$i(t) = i_L(t) - \frac{\frac{d}{dt} i_L(t)}{i_L(t)}$$

or

$$\frac{d}{dt} i_L(t) = i_L(t)[i_L(t) - i(t)] \tag{8.4}$$

The stationary equilibrium condition is obviously that the short rate equals the long rate $i_L(t) = i(t)$. We can linearise the equation in 8.4 at the steady state to get

$$\frac{d}{dt} i_L(t) = \alpha [i_L(t) - i(t)] \tag{8.5}$$

where $\alpha > 0$ is the steady-state value of the short and long nominal interest rate.

Just as we did for the simple monetary model, we can now analyse the response of the long rate to news about the future behaviour of short rates. This is left as an exercise for the reader.

We next consider the risk-neutral portfolio choice of an investor choosing between an index-linked consol and a one-period nominal bond.

An index linked consol pays a constant real coupon, $\gamma > 0$ each period until Kingdom Come. The nominal value of the coupon payment in period t is therefore $P(t)\gamma$, where P is the general price level.

Let P_γ be the money price of a consol. The total return on $1£$ invested in an index-linked consol for 1 period is$\frac{1}{P_\gamma(t)}\left(P(t)\gamma + P_\gamma(t+1)\right)$. Short nominal bonds and index-linked consols will only be held by risk-neutral speculators if

$$1 + i(t) = \frac{1}{P_\gamma(t)}\left(P(t)\gamma + E_t P_\gamma(t+1)\right)$$

In continuous time this becomes

$$i(t) = \frac{P(t)\gamma}{P_\gamma(t)} + \frac{E_t \dot{P}_\gamma(t)}{P_\gamma(t)} \tag{8.6}$$

We define the real price of an index-linked consol as $p_\gamma(t) \equiv \frac{P_\gamma(t)}{P(t)}$ This permits us to rewrite 8.6 as

$$r(t) \equiv i(t) - \frac{E_t \dot{P}(t)}{P(t)} = \frac{\gamma}{p_\gamma(t)} + \frac{E_t \dot{p}_\gamma(t)}{p_\gamma(t)}$$

Integrating this forward, dropping the expectation operator and imposing the usual 'no-bubble' condition yields

$$p_\gamma(t) = \gamma \int_t^\infty e^{-\int_t^s r(u)du} ds \tag{8.7}$$

Defining the long real interest rate, r_L,as the coupon yield on an index-linked consol, that is

$$r_L = \frac{\gamma}{p_\gamma}$$

we derive the following relationship between the long real rate and expected future short real rates:

$$r_L(t) = \left(\int_t^{\infty} e^{-\int_t^s r(u)du} ds \right)^{-1} \tag{8.8}$$

The long real rate is a forward-looking moving average of expected future short rates. We also have

$$r = r_L - \frac{\dot{r}_L}{r_L}$$

or

$$\dot{r}_L = r_L(r_L - r) \tag{8.9}$$

Linearising this around the steady state yields

$$\dot{r}_L = \alpha(r_L - r) \tag{8.10}$$

where $\alpha > 0$ is the steady-state value of the short and long real interest rate.

We now embed this relationship between short and long rates into the following simple AD-AS model

$$y = -\gamma r_L + f \tag{8.11}$$

$$m - p = ky - \lambda i \tag{8.12}$$

$$\dot{\pi} = \beta(y - \bar{y}) \tag{8.13}$$

$$\dot{r}_L = \alpha(r_L - r) \tag{8.14}$$

$$r \equiv i - \pi \tag{8.15}$$

$$\pi \equiv \dot{p} \tag{8.16}$$

$$\mu \equiv \dot{m} \tag{8.17}$$

$$\ell \equiv m - p \tag{8.18}$$

The exogenous variable f in the IS curve is any demand shifter, such as an expansion of government spending or a tax cut. All coefficients are positive.

A nominal interest rate rule

We consider the case where the authorities peg the short nominal interest rate.

The state-space representation of the model is

$$\begin{bmatrix} \dot{r}_L \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} \alpha & \alpha \\ -\beta\gamma & 0 \end{bmatrix} \begin{bmatrix} r_L \\ \pi \end{bmatrix} + \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & \beta & -\beta \end{bmatrix} \begin{bmatrix} i \\ f \\ \bar{y} \end{bmatrix}$$

The state matrix, $A \equiv \begin{bmatrix} \alpha & \alpha \\ -\beta\gamma & 0 \end{bmatrix}$, has a positive determinant and a positive trace. Both eigenvalues therefore have positive real parts. The model has one predetermined state-variable, π, and one non-predetermined state variable, r_L. For there to exist a unique continuously convergent solution, we need one stable and one unstable eigenvalue for A, that is, we want a saddlepoint configuration. Instead the model is completely unstable.

We now replace the exogenous short nominal interest rate by the following Taylor rule

$$i = \bar{i} + \delta_1 \pi + \delta_2 y$$

With the addition of the Taylor rule, the state-space representation of the model becomes

$$\begin{bmatrix} \dot{r}_L \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} \alpha(1+\delta_2\gamma) & \alpha(1-\delta_1) \\ -\beta\gamma & 0 \end{bmatrix} \begin{bmatrix} r_L \\ \pi \end{bmatrix} + \begin{bmatrix} -\alpha & \alpha\delta_2 & 0 \\ 0 & \beta & -\beta \end{bmatrix} \begin{bmatrix} \bar{i} \\ f \\ \bar{y} \end{bmatrix}$$

We will have a saddlepoint equilibrium if and only if $\beta\gamma\alpha(1-\delta_1) < 0$. This requires $\delta_1 > 1$: a higher rate of inflation leads to an increase in the nominal interest rate that exceeds the increase in the rate of inflation. The short real rate therefore rises. Note that the feedback coefficient of the short nominal rate on real output plays no role in ensuring saddle-point stability. For simplicity we therefore set $\delta_2 = 0$.

Steady state The steady state equilibrium condition are:

$$y = \bar{y} \tag{8.19}$$

$$r_L = r = \gamma^{-1}(f - \bar{y}) \tag{8.20}$$

$$\pi = \mu = (1-\delta_1)^{-1}\left(\bar{i} - r\right) = (1-\delta_1)^{-1}[\bar{i} + \gamma^{-1}(\bar{y} - f)] \tag{8.21}$$

$$i = \left(\frac{1}{1-\delta_1}\right)\bar{i} + \left(\frac{\delta_1}{1-\delta_1}\right)\gamma^{-1}(\bar{y} - f) \tag{8.22}$$

$$\ell = -\lambda\left(\frac{1}{1-\delta_1}\right)\bar{i} + \left[k - \lambda\left(\frac{\delta_1}{1-\delta_1}\right)\gamma^{-1}\right]\bar{y} + \lambda\left(\frac{\delta_1}{1-\delta_1}\right)\gamma^{-1}f \tag{8.23}$$

In the long run, real output equals its exogenous and constant capacity level. The short and long real interest rates are decreasing in capacity output and increasing in the fiscal stimulus. The inflation rate equals the growth rate of the nominal money stock, which is endogenous also in the long run. The coefficient relating the nominal interest rate to the rate of inflation, $\delta_1 > 0$ plays a key role in determining the steady state values of the nominal interest rate and the rate of inflation. A *lower* rate of steady-state inflation requires a *higher* value of $\bar{i}$, the exogenous component of the nominal interest rate rule. The intuition is provided by the dynamics of the response to an

increase in $\bar{\imath}$. An increase in $\bar{\imath}$ will, for a given rate of inflation, raise the short nominal rate, the short real rate and the long real rate and start a process of disinflation. As the inflation rate falls, the nominal interest rate responds more than one-for-one (according to the Taylor rule) and the short real rate will begin to decline again. In the long run, the inflation rate and the nominal interest rate are both lower. A fiscal expansion, in addition to raising long-run real rates, will raise the nominal interest rate and the rate of inflation and monetary growth in the long run. The mechanism again reflect the positive effect on inflation of a fiscal expansion in the short run, which, through the Taylor rule, leads to an increase in the nominal interest rate.

The response to shocks We continue to assume that $\delta_2 = 0$.

The r_L isocline is upward-sloping in r_L, π space:

$$: \qquad r_L|_{\dot{r}_L=0} = \bar{\imath} + (\delta_1 - 1)\pi$$

The π isocline is horizontal in r_L, π space:

$$\dot{\pi} = 0 : \qquad r_L|_{\dot{\pi}=0} = \gamma^{-1}(f - \bar{y})$$

The is a unique continuously convergent saddlepath, SS' that reaches the steady state, Ω. All solution trajectories that do not lie on SS' will asymptote to UU'. The equilibrium configuration corresponding to constant values of the exogenous variables is shown in Figure 8.1.

Figure 8.1 here

(1) An unanticipated, immediate and permanent increase in $\bar{\imath}$

The long-run real interest rate is unchanged. The long-run inflation rate falls. In Figure 8.2, the r_L isocline shifts up vertically by the amount of the increase in $\bar{\imath}$.

Figure 8.2 here

The economy is at Ω_1 before the shock. On impact, the inflation rate, π, is predetermined. The long rate jumps immediately toΩ_{12}, the point on the convergent saddlepath through the new steady state, Ω_2 vertically above Ω_1. From there on the long rate and the rate of inflation smoothly decline

towards Ω_2. Note that, from t_0 on, the long rate is falling, so the short real rate is above the long real rate. The nominal interest rate rises on impact by the same amount as $\bar{i}$. As the inflation rate falls. so does the nominal interest rate. Output is below its capacity level throughout.

(2) An unanticipated announcement of a future permanent increase in $\bar{i}$

As in the previous exercise, the long-run real interest rate is unchanged. The long-run inflation rate falls. In Figure 8.3, the r_L isocline shifts up vertically by the amount of the increase in $\bar{i}$.

Figure 8.3 here

The economy is at Ω_1 before the shock. On impact, when the future increase in $\bar{i}$ is announced, the inflation rate, π, is predetermined. The long rate jumps immediately toΩ_{12}, the point on that divergent solution path, drawn with reference to the initial steady state configuration, which will put it on the convergent saddlepath through the new steady state at t_1, when the increase in the nominal interest rate is actually implemented. At t_1 the system is at Ω_{13}. From t_1 on, the long rate and the rate of inflation decline smoothly towards Ω_2. Note that, between t_0 and t_1 the long rate is rising, so the long real rate is above the short real rate. After t_1, the positions of the long real rate and the short real rate are reversed.

Also, between t_0 and t_1, the short nominal rate is falling, since $\bar{i}$ has not yet been raised and the inflation rate is falling. Output is below its capacity level throughout.

(3) An unanticipated, immediate and permanent fiscal expansion

A permanent increase in f raises the steady state real interest rate and inflation rate. The π isocline shifts up in Figure 8.4.

Figure 8.4 here

If you were to draw the IS-LM diagram in i, y space at t_0, the IS curve which is vertical (because it depends on the long real rate, not on the short nominal or real rate) shifts to the right (holding constant r_L). The LM curve is horizontal, because of the Taylor rule. The future fiscal expansion raises future output. This gradually raises the future inflation rate. Through the Taylor rule, the nominal interest rate rises as inflation rises, and by more than the increase in the inflation rate. Future expected short real rates

therefore rise gradually for $t > t_0$. The long real rate is effectively a forward-looking weighted average of future short real rates, so it jumps up. It jumps immediately onto the convergent saddlepath through the new steady state, that is, to Ω_{12}. As the inflation rate rises, the nominal interest rate increases because of the Taylor rule. Output is everywhere about its capacity level.

(4) An unanticipated announcement of a future permanent fiscal expansion: contractionary fiscal expansions

As in the previous exercise, a permanent increase in f raises the steady state real interest rate and inflation rate. The π isocline shifts up in Figure 8.5.

Figure 8.5 here

Anticipating the future shift to the right of the IS curve and the associated future higher short real rates, the long real rate immediately increases at the announcement date. There is no actual fiscal expansion until $t_1 > t_0$, however, so the economy contracts, as the high long real rate depresses demand. When the announced fiscal expansion is finally implemented, output does expand. Note that r_L cannot jump at t_1. There is a recession between the announcement date and the implementation date and a boom following the implementation date. The nominal interest rate first falls, as inflation falls, and then rises, as inflation rises. Since the nominal interest rates responds more than one-for-one to the inflation rate (according to the Taylor rule), the short real rate first falls and then rises. The short real rate is below the long real rate throughout.

A monetary growth rule

$$\begin{bmatrix} \dot{r}_L \\ \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \begin{bmatrix} \alpha(1+\lambda^{-1}k\gamma) & \alpha & \alpha\lambda^{-1} \\ -\beta\gamma & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} r_L \\ \pi \\ \ell \end{bmatrix} + \begin{bmatrix} -\alpha\lambda^{-1}k & 0 & 0 \\ \beta & -\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ \bar{y} \\ \mu \end{bmatrix}$$

There are 2 predetermined state variables, π and ℓ and one non-predetermined state variable,r_L. In order for there to be a saddlepoint equilibrium, we need two stable eigenvalues and one unstable eigenvalue. The characteristic equation of the state matrix is

$$\chi(\rho) = \rho^3 - \alpha(1+\lambda^{-1}k\gamma)\rho^2 + \alpha\beta\gamma\rho - \alpha\beta\gamma\lambda^{-1} = 0 \tag{8.24}$$

Note that

$$\chi(0) = -\alpha\beta\gamma\lambda^{-1} < 0$$

$$\chi'(\rho) = 3\rho^2 - 2\alpha(1 + \lambda^{-1}k\gamma)\rho + \alpha\beta\gamma$$
$$\chi'(0) = \alpha\beta\gamma > 0$$

$$\chi''(\rho) = 6\rho - 2\alpha(1 + \lambda^{-1}k\gamma)$$
$$\chi''(0) = -2\alpha(1 + \lambda^{-1}k\gamma) < 0$$

For saddlepoint stability we require that there be two stable roots (with negative real parts) and one unstable (positive) root. Since $\chi(0) = -\alpha\beta\gamma\lambda^{-1} < 0$ [1], $\chi'(0) = \alpha\beta\gamma > 0$ and $\chi''(0) = -2\alpha(1+\lambda^{-1}k\gamma) < 0$ [2] we have three positive roots, as should be apparent from Figure 8.6.

Figure 8.6 here

The system is completely unstable.

We try the following Taylor rule to stabilise the system

$$\mu = \bar{\mu} + \delta_1\pi + \delta_2 y$$

The presumption is that $\delta_1 < 0$ and $\delta_2 < 0$.

The equations of motion become

$$\begin{bmatrix} \dot{r}_L \\ \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \begin{bmatrix} \alpha(1 + \lambda^{-1}k\gamma) & \alpha & \alpha\lambda^{-1} \\ -\beta\gamma & 0 & 0 \\ -\delta_2\gamma & \delta_1 - 1 & 0 \end{bmatrix} \begin{bmatrix} r_L \\ \pi \\ \ell \end{bmatrix} + \begin{bmatrix} -\alpha\lambda^{-1}k & 0 & 0 \\ \beta & -\beta & 0 \\ \delta_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ \bar{y} \\ \bar{\mu} \end{bmatrix}$$

The characteristic equation is

$$\begin{gathered} \chi(\rho) = \rho^3 - \alpha(1 + \lambda^{-1}k\gamma)\rho^2 + \alpha\gamma(\beta + \lambda^{-1}\delta_2)\rho - \alpha\beta\gamma\lambda^{-1}(1 - \delta_1) = 0 \\ \chi'(\rho) = 3\rho^2 - 2\alpha(1 + \lambda^{-1}k\gamma)\rho + \alpha\gamma(\beta + \lambda^{-1}\delta_2) \\ \chi''(\rho) = 6\rho - 2\alpha(1 + \lambda^{-1}k\gamma) \end{gathered} \tag{8.25}$$

[1] The determinant of the state matrix is $\alpha\beta\gamma\lambda^{-1} > 0$. The determinant of a matrix is the product of its eigenvalues. This is therefore consistent with either three unstable roots or two stable plus one unstable root.

[2] The trace of the state matrix, $\alpha(1 + \lambda^{-1}k\gamma) > 0$ is the sum of its eigenvalues. This is therefore implies that there is at least one unstable root.

It follows that

$$\chi(0) = -\alpha\beta\gamma\lambda^{-1}(1-\delta_1) < 0$$

$$\chi'(0) = \alpha\gamma(\beta + \lambda^{-1}\delta_2) \lesseqgtr 0$$

$$\chi''(0) = -2\alpha(1 + \lambda^{-1}k\gamma) < 0$$

Again, the last of these conditions implies that there are three unstable roots. The Taylor rule for money growth cannot stabilise the economic system. In order to stabilise the economic system when the growth rate of money is the instrument, the growth rate of nominal money must feed back, directly or indirectly, from the stock of real money balances, ℓ. It is easily verified that a rule such as

$$\mu = \bar{\mu} + \delta\ell$$

can stabilise the system:

$$\begin{bmatrix} \dot{r}_L \\ \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \begin{bmatrix} \alpha(1+\lambda^{-1}k\gamma) & \alpha & \alpha\lambda^{-1} \\ -\beta\gamma & 0 & 0 \\ 0 & -1 & \delta \end{bmatrix} \begin{bmatrix} r_L \\ \pi \\ \ell \end{bmatrix} + \begin{bmatrix} -\alpha\lambda^{-1}k & 0 & 0 \\ \beta & -\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ \bar{y} \\ \bar{\mu} \end{bmatrix}$$

$$\chi(\rho) = \rho^3 - [\delta + \alpha(1+\lambda^{-1}k\gamma)]\rho^2 + \alpha[\beta\gamma + (1+\lambda^{-1}k\gamma)\delta]\rho - \alpha\beta\gamma(\delta + \lambda^{-1}) = 0 \tag{8.26}$$

For instance, a sufficient condition for saddlepoint stability is a positive determinant and a negative trace of the state matrix. This is equivalent to the two following conditions:

$$\chi(0) = -\alpha\beta\gamma(\delta + \lambda^{-1}) < 0 \tag{8.27}$$

and

$$\chi''(0) = -2[\delta + \alpha(1 + \lambda^{-1}k\gamma)] > 0 \tag{8.28a}$$

It is clear that for 8.28a to be satisfied, δ must be sufficiently negative: $\delta < -\alpha(1 + \lambda^{-1}k\gamma)$. For 8.27 to be satisfied, however, δ cannot be too negative, since 8.27 implies $\delta > -\lambda^{-1}$.

We leave the analysis of the response of the system to shocks as an exercise for the reader.

8.2 The Stock Market

Useful references are Blanchard [2] and Hayashi [18].

Let $V(t)$ be the nominal value of the earnings or profits of enterprises. Assume all earnings are paid out as dividends. $Q(t)$ is the money value of corporate stocks or equity. The total nominal rate of return on 1£ invested in stocks is $\frac{V(t)+Q(t+1)}{Q(t)}$. Risk neutral speculators equate the expected rates of return on equity and short nominal debt:

$$\frac{V(t)+E_tQ(t+1)}{Q(t)} = 1+i(t) \tag{8.29}$$

In continuous time, the corresponding equilibrium condition is

$$\frac{V(t)+E_t\dot{Q}(t)}{Q(t)} = i(t)$$

Let $v(t) \equiv V(t)/P(t)$ and $q(t) \equiv Q(t)/P(t)$

We can rewrite 8.29 as follows (the length of the unit period is $\Delta t > 0$) :

$$\frac{\frac{V(t)\Delta t}{P(t+\Delta t)} + \frac{Q(t)}{P(t)} + E_t\left(\frac{Q(t+\Delta t)}{P(t+\Delta t)} - \frac{Q(t)}{P(t)}\right)}{\frac{Q(t)}{P(t)}} = (1+i(t)\Delta t)\,\frac{P(t)}{P(t+\Delta t)}$$

In the limit as $\Delta t \to 0$ (and making some regularity assumptions like $lim_{\Delta t \to 0} E_tP(t+\Delta t) = P(t)$), this reduces to

$$\frac{v(t)+E_t\dot{q}(t)}{q(t)} = r(t)$$

Integrating this forward in time and dropping the expectation operator produces

$$q(t) = \int_t^{\infty} e^{-\int_t^s r(u)du} v(s)ds + ke^{\int_{t_0}^t r(u)de}$$

where k is an arbitrary constant. We choose the bubble-free solution corresponding to $k=0$. This implies

$$q(t) = \int_t^{\infty} e^{-\int_t^s r(u)du} v(s)ds \tag{8.30}$$

The real value of the stock market is the present discounted value of all future real corporate earnings. Note that equation 8.30 looks very much like the pricing equation for long real bonds given in 8.7. The only difference is that with index-linked consols, the real coupon payment stream is constant, which the earnings of the firm, v, can be assumed to depend on the state of the economy.

Note that this stock market valuation function works also for an endowment economy, i.e. an economy without capital accumulation. Let output, y, be produced by labour only, with a positive but diminishing marginal product of labour.

$$
\begin{aligned}
y &= F(L) \\
L &\geq 0 \\
F(0) &= 0 \\
F' &> 0 \\
F'' &< 0 \\
\lim_{L\to 0} F'(L) &= \lim_{L\to\infty} \frac{1}{F'(L)} = 0
\end{aligned}
$$

It follows that

$$v = y - wL$$

where w is the real wage. Assume the real wage equals the marginal product of labour,

$$w = F'(L)$$

It follows that

$$
\begin{aligned}
v &= v(y) \\
v' &= -LF''(L)\,(F'(L))^{-1} > 0
\end{aligned}
$$

We now embed this 'Tobin's q' relation into the same simple AD-AS framework used to study the term structure of interest rates

model

$$y = \gamma q + f \tag{8.31}$$

$$m - p = ky - \lambda i \tag{8.32}$$

$$\dot{\pi} = \beta(y - \bar{y}) \tag{8.33}$$

$$\dot{q} = rq - \upsilon \tag{8.34}$$

$$\upsilon = \upsilon(y) \tag{8.35}$$

$$\upsilon' > 0$$

$$r \equiv i - \pi \tag{8.36}$$

$$\pi \equiv \dot{p} \tag{8.37}$$

$$\mu \equiv \dot{m} \tag{8.38}$$

$$\ell \equiv m - p \tag{8.39}$$

Equation 8.31 makes aggregate demand an increasing function of q. This can reflect both a wealth effect on private consumption and a positive effect of a higher value of q on private investment. Equation 8.35 makes profits an increasing function of output.

8.2.1 A nominal interest rate rule

We assume again that the short nominal interest rate is governed by the following Taylor rule

$$i = \bar{i} + \delta_1 \pi + \delta_2 y$$

The model again has one predetermined state variable, π, and one non-predetermined state variable, q. The two boundary conditions are an initial condition for π and 'converge if you can'.

The equations of motion for the state variables are

$$\dot{q} = [\bar{\imath} + \delta_2 f + (\delta_1 - 1)\pi + \delta_2 \gamma q]q - \upsilon(\gamma q + f)$$

$$\dot{\pi} = \beta\gamma q + \beta(f - \bar{y})$$

Linearising this at the steady state gives

$$\begin{bmatrix} \dot{q} \\ \dot{\pi} \end{bmatrix} \approx \begin{bmatrix} \bar{\imath} + \delta_2 f + (\delta_1 - 1)\pi + 2\delta_2\gamma q - \gamma\upsilon' & q(\delta_1 - 1) \\ \beta\gamma & 0 \end{bmatrix} \begin{bmatrix} q - \bar{q} \\ \pi - \bar{\pi} \end{bmatrix} + \begin{bmatrix} q & \delta_2 q - \upsilon' & 0 \\ 0 & \beta & -\beta \end{bmatrix} \begin{bmatrix} \bar{\imath} - \bar{\bar{\imath}} \\ f - \bar{f} \\ \bar{y} - \bar{\bar{y}} \end{bmatrix} \tag{8.40}$$

Steady State

The steady state equilibrium conditions are:

$$y = \bar{y} \tag{8.41}$$

$$q = \gamma^{-1}(\bar{y} - f) \tag{8.42}$$

$$r = \frac{\gamma\upsilon(\bar{y})}{\bar{y} - f} \tag{8.43}$$

$$\pi = \mu = \left(\frac{1}{1-\delta_1}\right)\bar{\bar{\imath}} + \left(\frac{\delta_2}{1-\delta_1}\right)\bar{y} + \left(\frac{1}{\delta_1 - 1}\right)\left(\frac{\gamma\upsilon(\bar{y})}{\bar{y}-f}\right) \tag{8.44}$$

$$i = \left(\frac{1}{1-\delta_1}\right)\bar{\bar{\imath}} + \left(\frac{\delta_2}{1-\delta_1}\right)\bar{y} + \left(\frac{\delta_1}{\delta_1 - 1}\right)\left(\frac{\gamma\upsilon(\bar{y})}{\bar{y}-f}\right) \tag{8.45}$$

$$\ell = -\lambda\left(\frac{1}{1-\delta_1}\right)\bar{\bar{\imath}} + \left[k - \lambda\left(\frac{\delta_2}{1-\delta_1}\right)\right]\bar{y} - \lambda\left(\frac{\delta_1}{1-\delta_1}\right)\left(\frac{\gamma\upsilon(\bar{y})}{\bar{y}-f}\right) \tag{8.46}$$

Note that for the economy to be viable, $\upsilon(\bar{y}) > 0$ and $\bar{y} - f > 0$.

In the long run, the real value of the stock market is independent of monetary policy. More expansionary fiscal policy raises the long-run real interest rate. Since real profits are constant in the long run, the discounted value of real profits falls. Because of the Taylor rule for the nominal interest rate, expansionary fiscal policy raises both the nominal interest rate and the rate of inflation in the long run, but it raises the nominal interest rate by more than the rate of inflation.

The Response To Shocks

The determinant of the state matrix of 8.40 is $(1-\delta_1)q\beta\gamma$. For this system to have a saddlepoint configuration we therefore require, as before, that $\delta_1 > 1$. Again, δ_2 does not affect the conditions for existence of a saddlepoint equilibrium (although it does affect the steady-state of the system and its response to shocks). We therefore again set $\delta_2 = 0$.The linearised dynamic system now simplifies to:

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{\pi} \end{bmatrix} \approx \begin{bmatrix} \Omega & q(\delta_1 - 1) \\ \beta\gamma & 0 \end{bmatrix} \begin{bmatrix} q - \bar{q} \\ \pi - \bar{\pi} \end{bmatrix} \\ + \begin{bmatrix} q & -\upsilon' & 0 \\ 0 & \beta & -\beta \end{bmatrix} \begin{bmatrix} \bar{\imath} - \bar{\bar{\imath}} \\ f - \bar{f} \\ \bar{y} - \bar{\bar{y}} \end{bmatrix} \end{aligned} \tag{8.47}$$

$$\Omega = \bar{\imath} + (\delta_1 - 1)\pi - \gamma\upsilon'$$

The q isocline is given by

$$q-\bar{q}|_{\dot{q}=0}=\frac{(1-\delta_1)\,q}{\Omega}(\pi-\bar{\pi})-\frac{q}{\Omega}(\bar{\imath}-\bar{\bar{\imath}})+\frac{\upsilon'}{\Omega}(f-\bar{f})$$

The π isocline is given by

$$q-\bar{q}|_{\dot{\pi}=0}=\gamma^{-1}[\bar{y}-\bar{\bar{y}}-(f-\bar{f})]$$

The slope of the q isocline is ambiguous, as it depends on the sign of $\Omega=r-\gamma\upsilon'(y)=\frac{\partial\dot{q}}{\partial q}$ (see 8.31, 8.35 and 8.34). Consider first the case where $\Omega<0$. This will be the case if profits respond strongly to output and stock market valuations have a strong effect on demand. When $\Omega<0$,the q isocline is upward-sloping. The equilibrium configuration is shown in Figure 8.7.

Figure 8.7 here

When $\Omega>0$, the q isocline is downward-sloping. This equilibrium configuration is shown in Figure 8.8.

Figure 8.8 here

Restrictive monetary policy We consider the effect of an unanticipated, immediate and permanent increase in $\bar{\imath}$. In the short run, this amounts to an increase in the short-run nominal rate of interest. In the long run, the nominal rate will decline, as the rate of inflation falls and the Taylor rule prompts nominal rate reductions that exceed the decline in the inflation rate.

The response to the monetary contraction is very similar to that in the model with the long real interest rate in the previous section. The initial increase in the short nominal rate raises short real rates during the transition to the new steady state. This causes the stock market to fall immediately, just as it caused the long real rate of interest to rise in the previous section. The decline in the stock market is reinforced by the fact that output and therefore profits are lower during the transition. The stock market index is the present discounted value of future profits. The discount rates increase and the future profits decrease. In the long run, output is restored at its exogenous capacity level. The real interest rate is also independent of monetary policy in the long run, so in the long run the stock market returns to its original value (in real terms). The inflation rate and the nominal interest rate are lower.

This scenario applies both when $\Omega < 0$ (Figure 8.9) and when $\Omega > 0$ (Figure 8.10).

Figure 8.9 here

Figure 8.10 here

Expansionary fiscal policy When there is an increase in f, the π isocline shifts down. The q isocline shifts down when $\Omega < 0$ (that is, when the q isocline is upward-sloping). It shifts up when $\Omega > 0$.

It appears that, when we increase f, it is ambiguous whether the stable manifold, SS' shifts up or down, since both the q isocline and the π isocline shift.

Using the results of Chapter 6, we can establish that the equation for the stable manifold is [3],

$$
\begin{aligned}
q - \bar{q}|_{SS'} &= \tfrac{-\rho_2}{\beta\gamma}(\pi - \bar{\pi}) \\
&- \left(\tfrac{\rho_2-\rho_1}{q(\delta_1-1)\rho_2}\right)\left[\begin{array}{l} \tfrac{-\rho_2}{\rho_1-\rho_2}q(\bar{\imath} - \bar{\bar{i}}) + \left(\left(\tfrac{\rho_2}{\rho_1-\rho_2}\right)\upsilon' + \tfrac{q(\delta_1-1)}{\rho_2-\rho_1}\beta\right)(f - \bar{f}) \\ -\tfrac{q(\delta_1-1)}{\rho_2-\rho_1}\beta(\bar{y} - \overline{\bar{y}}) \end{array}\right]
\end{aligned}
$$

where $\rho_1 < 0$ is the stable eigenvalue and $\rho_2 > 0$ the unstable eigenvalue of the state matrix.

If follows, since $\rho_1 < 0$ and $\rho_2 > 0$ that the direction in which the stable manifold shifts when f increases depends on the sign of $\Psi = \left(\frac{\rho_2}{\rho_1-\rho_2}\right)\upsilon' + \frac{q(\delta_1-1)}{\rho_2-\rho_1}\beta$. If $\Psi > 0$, the stable manifold shifts down when f increases. This is the case drawn in Figure 8.11. If $\Psi < 0$, it shifts up. This is the case drawn in Figure 8.12.

Figure 8.11 here

Figure 8.12 here

[3] We use the fact that $\rho_1\_\Omega < 0$. This follows because Ω is the trace of the state matrix, so $\Omega = \rho_1 + \rho_2$, $\rho_1 - \Omega = -\rho_2 < 0$.

Following Blanchard [2] we can refer to the $\Psi > 0$ case as the 'bad news' case and to the $\Psi < 0$ case as the 'good news' case. The news in question refers to the impact of news about a fiscal expansion on the stock market. In this long-run full employment model, expansionary fiscal policy is always bad news for the stock market. Long-run output and profits are unchanged but the long-run real interest rate increases. In the short run however, expansionary fiscal policy raises output above its capacity level and this will raise profits. When $\Psi < 0$, the increase in transitional profits dominate, initially, the effect of the higher real interest rates discounting these higher future profits. When $\Psi > 0$, the higher real interest rates dominate the higher transitional profits. Note from Figures 8.11 and 8.12 that, whatever happens on impact to the stock market, the economy will go through a boom period with rising inflation following an unanticipated immediate and permanent fiscal expansion. In the bad news case, the stock market falls at the announcement date, which in the good news case it rises on impact. You should be able to convince yourself that both the good news and the bad news scenarios can occur when $\Omega > 0$ and the q isocline is downward-sloping.

Contractionary expansionary fiscal policy again Consider the case where the bad news scenario prevails, that is, $\Psi > 0$. When there is an unexpected announcement, at $t = t_0$ of a future permanent fiscal expansion, starting at $t = t_1 > t_0$, the economy will go into recession between the announcement date, t_0 and the implementation date, t_1. Once the fiscal expansion is implemented, the economy goes into a boom period. The bad news scenario is very similar to the scenario analysed in the previous section, when the announcement of a future fiscal expansion caused an immediate increase in the long real rate of interest. In the present model, the stock market falls on the announcement date. Anticipated higher short real rates dominate anticipated higher transitional profits once the fiscal expansion gets under way. The behaviour of the stock market and the inflation rate following the announcement is shown in Figure 8.13.

Figure 8.13 here

When the future fiscal expansion is announced, the stock market drops from Ω_1 to Ω_{12}, onto that divergent trajectory, drawn with reference to the initial steady state, Ω_1, that will put it on the stable manifold through the new steady state, Ω_1, when the increase in fiscal spending is actually implemented (at Ω_{13}). The economy is in recession, with inflation falling, between t_0 and t_1. It is in a boom, with inflation rising, from t_1 on.

8.2.2 A Monetary Growth Rule

When the growth rate of the nominal money stock is the instrument of the monetary authorities, we have a three-dimensional dynamic system:

$$\dot{q} = \left(-\lambda^{-1}\ell + \lambda^{-1}k(\gamma q + f)\right) q - \pi q - \upsilon(\gamma q + f)$$

$$\dot{\pi} = \beta\gamma q + \beta(f - \bar{y})$$

$$\dot{\ell} = \mu - \pi$$

It can be verified that an 'augmented Taylor/McCallum rule' for money growth, such as

$$\mu = \bar{\mu} + \delta_1\pi + \delta_2 y + \delta_3\ell$$

can ensure that the linearised system has the desired saddlepoint properties, with two stable eigenvalues, corresponding to the two predetermined state variables π and ℓ and one unstable eigenvalue, corresponding to the non-predetermined state variable q.

It will again turn out to be the case that feedback from inflation and output will not stabilise the system, but feedback from the stock of real money balances may do so.

8.3 A Forward-Looking Foreign Exchange Market

Useful references are Dornbusch [13], Buiter and Miller [9] and Blanchard and Fischer [3]. A domestic financial investor can invest 1 £ in one-period domestic securities with a known, safe rate of interest $i(t)$. In period $t+1$ he will therefore receive $1 + i(t)$ *£s* for certain. He can also convert his 1 £ into *Euros* in the spot foreign exchange market, at the spot nominal exchange rate $S(t)$. The spot price of foreign exchange is the number of *£s* that has to be paid at time t in exchange for 1 *Euro* to be delivered at time t. He can invest his $\frac{1}{S(t)}$ *Euros* at the safe (in terms of *Euros*) one-period interest rate, $i^*(t)$, .giving him $\frac{1}{S(t)}(1 + i^*(t))$ *Euros* for sure in period $t+1$. In the

forward exchange market, he can, in period t, agree a contract to sell these $\frac{1}{S(t)}(1+i^*(t))$ *Euros* in period $t+1$,for a price, agreed at time t, of $F(t,t+1)$ *£s* per *Euro*. In general, $F(t,t+N)$ is the *N-period forward (nominal) exchange rate*, that is the £ price of a *Euro*, agreed at time t, for delivery at time $t+N$. This means that he will again end up with a known amount of *£s* in period $t+1$, equal to $\frac{1}{S(t)}(1+i^*(t))F(t,t+1)$. If there are no transactions costs and the prices are independent of the quantities sold or bought by an individual trader, these two routes for investing 1*£* in period t and ending up with a known amount of *£s* in period $t+1$, should have the same return. Otherwise there would be a money machine. The first implication of the assumption that there are no riskless pure profits, is therefore that

$$1+i(t)=\frac{1}{S(t)}(1+i^*(t))F(t,t+1) \tag{8.48}$$

Condition 8.48 is known as *Covered Interest Parity* or CIP.

If there are risk-neutral speculators (and if Jensen's inequality does not bother us if there is inflation uncertainty), it will be the case that the forward rate equals the current expectation of the future spot rate, or

$$F(t,t+1)=E_tS(t+1) \tag{8.49}$$

Combining 8.48 and 8.49, yields the *Uncovered Interest Parity* Condition, or UIP:

$$1+i(t)=\frac{1}{S(t)}(1+i^*(t))E_tS(t+1) \tag{8.50}$$

In continuous time, UIP becomes [4]

---

[4]From $1+i(t)\Delta t=\frac{1}{S(t)}(1+i^*(t)\Delta t)F(t,t+\Delta t)$, where $\Delta t>0$ is the length of the unit period, it follows, provided that $\lim_{\Delta t\to 0}F(t,t+\Delta t)=F(t,t)=S(t)$, that

$$\frac{1}{F(t,t)}\frac{\partial F(t,t)}{\partial t}=i(t)-i^*(t)$$

Where $\frac{\partial F(t,t)}{\partial t}\equiv\lim_{\Delta t\to 0}\frac{F(t,t+\Delta t)-F(t,t)}{\Delta t}$
If we assume that

$$E_t\dot{S}(t)\equiv E_t\lim_{\Delta t\to 0}\frac{S(t+\Delta t)-S(t)}{\Delta t}=\frac{\partial F(t,t)}{\partial t}$$

, 8.51 follows.

$$i(t) = i^*(t) + \frac{E_t \dot{S}(t)}{S(t)} \tag{8.51}$$

Letting $s \equiv \ln S$, this can be rewritten as

$$i(t) = i^*(t) + E_t \dot{s}(t) \tag{8.52}$$

Integrating 8.52 forward in time gives

$$s(t) = E_t \int_t^{\infty} [i^*(v) - i(v)]dv + E_t s(\infty) \tag{8.53}$$

Thus, the current spot nominal exchange rate increases (spot £ weakens), if future *Euro* nominal interest rates rise relative to future £ nominal interest rates, or if the expected long-run equilibrium nominal spot rate $s(\infty)$ weakens (increases).

Risk premia, tax wedges, transactions costs and departures from competitive efficiency, can cause the UIP relationship to fail. Let $\rho(t)$ be the excess of the period t nominal interest rate over its UIP value. It follows that,

$$i(t) = i^*(t) + E_t \dot{s}(t) + \rho(t) \tag{8.54}$$

This implies that, for any $T \geq t$ (including the limit as $T \to \infty$),

$$s(t) = E_t \int_t^T [i^*(v) - i(v)]dv + E_t \int_t^T \rho(v)dv + E_t s(T) \tag{8.55}$$

Thus the change in the nominal spot exchange rate between periods t_0 and $t > t_0$ is given by

$$\begin{aligned} s(t) - s(t_0) = {} & (E_t - E_{t_0}) \textstyle\int_t^T [i^*(v) - i(v)]dv - E_{t_0} \int_{t_0}^t [i^*(v) - i(v)]dv \\ & + (E_t - E_{t_0}) \textstyle\int_t^T \rho(v)dv - E_{t_0} \int_{t_0}^t \rho(v)dv \\ & + (E_t - E_{t_0}) s(T) \end{aligned} \tag{8.56}$$

$E_t - E_{t_0}$ is the revision in expectations between time t_0 and time t.

Unless we can impose some structure on $\rho(t)$, this is no more than a decomposition, with $\rho(v)$ taking the role of the residual that ensures a perfect

fit. Of course, even if we observed $\rho(v)$ directly, we still would have to make estimates of unobserved expectations of future interest rate differentials and of the unobserved expectation of the terminal nominal spot rate. For simplicity I assume in the rest of this section that $\rho = 0$.

Let p be the logarithm of the domestic (UK) price level and p^* the logarithm of the Euroland price level. Also, $\pi \equiv \dot{p}$ is the UK rate of inflation, $\pi^* \equiv \dot{p}^*$ is the Euroland rate of inflation, $r \equiv i - \pi$ is the UK short real interest rate and $r^* \equiv i^* - \pi^*$ is the Euroland short real interest rate. Finally, $c \equiv s + p^* - p$ is the UK real exchange rate, the relative price of imports to exports, or a measure of UK competitiveness relative to Euroland.

From UIP it follows that

$$r(t) = r^*(t) + E_t\dot{c}(t) \tag{8.57}$$

Integrating 8.57 forward in time gives

$$c(t) = E_t \int_t^{\infty} [r^*(s) - r(s)]ds + E_t c(\infty) \tag{8.58}$$

Thus, the current spot real exchange rate increases (UK competitiveness improves or the terms of trade worsen), if future *Euro* real interest rates rise relative to future £ real interest rates, or if the expected long-run equilibrium real spot rate $c(\infty)$ weakens (increases).

We embed the UIP relationship in a simple open economy AD/AS model with a floating exchange rate. All coefficients are positive. The IS curve, 8.59, contains an exogenous demand shifter, f, which can represent expansionary fiscal policy shocks, private domestic demand shocks or net trade shocks, such as a shock to world GDP or world trade.

$$y = -\gamma r + \eta c + f \tag{8.59}$$

$$m - p = ky - \lambda i \tag{8.60}$$

[5]

[5] As Stephen Turnovsky points out, the demand for real money balances depends on real income and both real income and real money balances are defined in terms of the consumer price index $\tilde{p}$. Let p be the GDP deflator. Assume that domestic income consists

$$i = i^* + \dot{s} \tag{8.61}$$

$$\dot{\pi} = \beta(y - \bar{y}) \tag{8.62}$$

$$c \equiv s + p^* - p \tag{8.63}$$

$$m - p \equiv \ell \tag{8.64}$$

$$\dot{m} \equiv \mu$$

8.3.1 A Nominal Interest Rate Rule

We again consider a Taylor rule for the nominal interest rate, but now add the real exchange rate as an argument in the monetary reaction function:

$$i = \bar{i} + \delta_1 \pi + \delta_2 y + \delta_3 c \tag{8.65}$$

Note that π, the inflation rate of the GDP deflator, can in this simple model be interpreted as the 'domestically generated inflation rate', or DGI. DeAnne Julius and John Vickers, both of the MPC, have suggested that a

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of domestic GDP (no international diversification here). Monetary equilibrium would then be given by

$$m - \tilde{p} = k(y + p - \tilde{p}) - \lambda i$$

Only if $k = 1$ (money demand is homogeneous of degree 1 in the relevant scale variables (income and/or wealth), will we be able to write monetary equilibrium in terms of the GDP deflators alone. In general

$$m - p = ky + (1 - k)(\tilde{p} - p) - \lambda i$$

so the real exchange rate would directly enter the monetary equilibrium condition. I prefer the expositional simplicity of the specification adopted in the body of the chapter.

Taylor rule that feeds back from the cost of living index, that is, a price index such as the CPI in the USA or the RPI in the UK, might have rather different operating characteristics from a Taylor rule responding to DGI. The reason is that import prices, and thus the exchange rate, directly enter the CPI. Let $\tilde{P} \equiv P^{1-\alpha}(SP^*)^\alpha$, $0 \leq \alpha \leq 1$ be the consumer price index, which we assume to be the price index of a bundle of both domestically produced and imported goods and services. The coefficient α can be interpreted as the share of imports in the consumption bundle. Also, let $\tilde{p} = \ln \tilde{P}$ and $\tilde{\pi} = \frac{d}{dt}\tilde{p}$ be the inflation rate of the consumer price index. Since the exchange rate can make discrete jumps, this would make the instantaneous rate of consumer price inflation infinite whenever such discrete jumps in the exchange rate happen. We get around this analytical difficulty by assuming that the Taylor rule makes the short nominal rate respond to the *expected* rate of inflation. Since discrete jumps in the consumer price level are due to discrete jumps in the nominal exchange rate, and since speculative efficiency rules out anticipated discrete jumps in the nominal exchange rate, this eliminates the infinite nominal interest rate anomaly. The modified Taylor rule would be

$$i = \bar{i} + \delta_1 E_t \tilde{\pi} + \delta_2 y + \delta_3 c \tag{8.66}$$

Noting that $\tilde{\pi} = (1-\alpha)\pi + \alpha(\pi^* + \dot{s}) = \bar{\pi} + \alpha\dot{c}$, we can rewrite 8.66 as

$$i = \bar{i} + \delta_1 \pi + \delta_1 \alpha E_t \dot{c} + \delta_2 y + \delta_3 c \tag{8.67}$$

The state-space representation of the model under the DGI Taylor rule 8.65 is as follows:

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} \frac{\delta_3 + \delta_2 \eta + \gamma\delta_2\delta_3(1-\eta)}{1+\gamma\delta_2} & \frac{\delta_1 - 1}{1+\gamma\delta_2} \\ \frac{\beta\eta(1-\gamma\delta_3)}{1+\gamma\delta_2} & \frac{\beta\gamma(1-\delta_1)}{1+\gamma\delta_2} \end{bmatrix} \begin{bmatrix} c \\ \pi \end{bmatrix} + \begin{bmatrix} \frac{1+\delta_2(\gamma-1)}{1+\gamma\delta_2} & \frac{\delta_2}{1+\gamma\delta_2} & 0 & -1 & 1 \\ \frac{-\beta\gamma}{1+\gamma\delta_2} & \frac{\beta}{1+\gamma\delta_2} & -\beta & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{i} \\ f \\ \bar{y} \\ i^* \\ \pi^* \end{bmatrix} \tag{8.68}$$

The determinant of the state matrix, denoted D, is given by

$$D = (1-\delta_1)\left(\frac{1}{1+\gamma\delta_2}\right)^2 \beta\left[\eta + (1-\eta)\gamma\delta_3(1+\gamma\delta_2)\right] \tag{8.69}$$

The model has one predetermined state variable, π, and one non-predetermined state variable, c. The real exchange rate is non-predetermined because the nominal exchange rate is non-predetermined. Both domestic and foreign prices are assumed to be predetermined in their own respective currencies. For the system to have a saddlepoint configuration, we need D to be negative.

Note that, if $\delta_3 = 0$, that is, if there is no feedback response of the short nominal interest rate to the real exchange rate, then stability again requires $\delta_1 > 1$: a higher rate of inflation leads to a larger increase in the short nominal rate. In that case, feedback from real output cannot stabilise the system.

If $\delta_3 > 0$ (a depreciation of the real exchange rate leads to a rise in the nominal short rate and if $\delta_2 > 0$ (higher output leads to a higher nominal rate), $\delta_1 > 1$ remains necessary (and sufficient) for a saddlepoint configuration. If $\delta_3 < 0$, it is possible to have saddlepoint configurations, either with $\delta_1 > 1$(if $\eta + (1-\eta)\gamma\delta_3(1+\gamma\delta_2) > 0$) or with $\delta_1 < 1$ ((if $\eta + (1-\eta)\gamma\delta_3(1+\gamma\delta_2) < 0$).

For simplicity I will, in what follows, assume that $\delta_2 = \delta_3 = 0$. The model then simplifies to

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} 0 & \delta_1 - 1 \\ \beta\eta & \beta\gamma(1-\delta_1) \end{bmatrix} \begin{bmatrix} c \\ \pi \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -\beta\gamma & \beta & -\beta & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{i} \\ f \\ \bar{y} \\ i^* \\ \pi^* \end{bmatrix} \tag{8.70}$$

The condition for saddlepoint stability now is

$$\delta_1 > 1$$

Steady State

The steady state (constant values of the exogenous variables and $\dot{\pi} = \dot{c} = 0$) is characterised by

$$y = \bar{y} \tag{8.71}$$

$$r = r^* \tag{8.72}$$

$$\pi = \pi^* + \dot{s} = \mu = (1 - \delta_1)^{-1} (\bar{\imath} - r^*) \tag{8.73}$$

$$c = \eta^{-1}(\bar{y} - f) + \eta^{-1}\gamma r^* \tag{8.74}$$

$$i = \left(\frac{1}{1 - \delta_1}\right) \bar{\imath} - \left(\frac{\delta_1}{1 - \delta_1}\right) r^* \tag{8.75}$$

$$\ell = k\bar{y} - \left(\frac{\lambda}{1 - \delta_1}\right) \bar{\imath} + \left(\frac{\lambda\delta_1}{1 - \delta_1}\right) r^* \tag{8.76}$$

Real output is at its exogenous and constant capacity level in the long run. The long-run domestic real interest rate equals the long-run world real interest rate, because the long-run real exchange rate is constant over time. A lower level of capacity output and an increase in the fiscal stimulus cause a long-run appreciation of the real exchange rate (a decline in c, that is an increase in the relative price of domestic goods to foreign goods). A higher world real interest rate requires a long-run real depreciation in order to maintain balance between a given net supply of domestic goods and a lower level of interest-sensitive domestic spending.

What happens to inflation and the nominal interest rate in the long run reflects the operation of the Taylor rule with $\delta_1 > 1$. A higher value of the exogenous component in the nominal interest rate rule, $\bar{\imath}$, is associated with a lower long-run rate of inflation and a lower long-run nominal interest rate. Remember that, in the short run, an increase in $\bar{\imath}$ means an equal increase in i, that is, a more restrictive monetary policy. This causes a recession which gradually reduces inflation and with it the nominal interest rate. A higher world real interest rate raises the long-run rate of inflation because it causes a temporary boom (because of a depreciation of the real exchange rate).

Since $\dot{c} = 0$ in steady state, these steady state conditions apply both when the Taylor rule responds to DGI and when the Taylor rule responds to CPI inflation.

The Response to Shocks when the Taylor Rule Responds to DGI

The c isocline is given by

$$\pi|_{\dot{c}=0} = \frac{1}{1-\delta_1}(\bar{i} - i^* - \pi^*)$$

It is a vertical line in c, π space.

The π isocline is given by

$$c|_{\dot{\pi}=0} = \eta^{-1}\gamma(\delta_1 - 1)\pi + \eta^{-1}\gamma\bar{i} + \eta^{-1}(\bar{y} - f)$$

Because of the Taylor rule, this is upward-sloping in c, π space. The saddlepoint configuration is shown in Figure 8.14.

Figure 8.14 here

Restrictive monetary policy The economy starts in steady state, at Ω_1 in Figure 8.15. An unanticipated, immediate and permanent increase in $\bar{i}$ leads to the following dynamic response:

Figure 8.15 here

The c isocline shifts to the left and the π isocline shifts up. Note that we know that the steady state real exchange rate does not change. The impact effect of an unanticipated increase in $\bar{i}$ is to raise the short domestic nominal and real interest rates. We know the long-run real exchange rate is unaffected. The differential between domestic and foreign real interest rates therefore increases and the real exchange rate makes a discrete jump down when the monetary policy news hits. The resulting loss of competitiveness causes a decline in output. Inflation falls. During the adjustment path, the domestic short real interest rate is above the world real interest rate, so the real exchange rate depreciates gradually back to its initial level.

The initial jump down in the real exchange rate must be due to an initial jump down in the nominal exchange rate of the same magnitude, since the domestic price level is predetermined.

Expansionary fiscal policy An expected, immediate and permanent increase in f at t_0 reproduces exactly the response to any exogenous demand shock in the static Mundell-Fleming model with a floating exchange rate and free capital mobility. The π isocline shifts down. The c isocline stays put. The new steady state has a lower value of c, that is, a worse level of international competitiveness, and the same rate of inflation. The transition from

the initial steady state, Ω_1 to the new steady state, Ω_2, is instantaneous. There is a discrete appreciation of the nominal and real exchange rate at t_0. The net trade surplus falls by the same amount as the exogenous increase in demand. No other nominal or real variable changes. The 'dynamic' response is shown in Figure 8.16.

Figure 8.16 here

Contractionary fiscal expansions once more The unexpected announcement, at t_0, of a future permanent increase in f, starting at $t_1 > t_0$ leads to the dynamic response shown in Figure 8.17.

Figure 8.17 here

At the announcement date, t_0, the nominal and real exchange rate fall discretely from Ω_1 to Ω_{12}, onto that divergent trajectory, drawn with reference to the initial steady state, Ω_1, that will place it at the implementation date, t_1, on the convergent saddlepath S_2S_2' through the new steady state Ω_2. At t_0 the economy has experienced a loss of competitiveness, but the fiscal stimulus has not yet come through, so the economy goes into recession and the inflation rate falls. As the inflation rate falls, the nominal short interest rates falls also (and by more than the inflation rate), because of the Taylor rule. At $\Omega\ _{12}$, $\dot{c} = 0$, so the short real interest rate does not change immediately. After t_0, $\dot{c} < 0$, and $r = r^* + \dot{c}$, so the real interest rate falls, at an increasing rate until t_1, because c falls at an increasing rate until t_1. At t_1, there is a point of inflection in the path of c, and a discontinuity in $\dot{c}$. The real interest rate jumps up above its new (and old) steady state level, r^*, and from then on declines asymptotically back to r^* (note that $\dot{c} < 0$ also after t_1, although the rate of decline of c falls gradually towards zero. The cumulative increase in domestic short real rates after t_1 exceeds the cumulative decline in domestic short real rates between t_0 and t_1. This is the reason for the initial drop in the value of c.

When the fiscal expansion is actually implemented at t_1, output rises above its capacity level and the rate of inflation rises back towards its starting value. The short nominal interest rate rises more than one-for-one with the rate of inflation because of the Taylor rule.

The Response to Shocks when the Taylor Rule Responds to CPI Inflation

Now consider the behaviour of the system under the following CPI inflation version of the Taylor rule:

$$i = \bar{i} + \delta_1 \pi + \delta_1 \alpha E_t \dot{c} \tag{8.77}$$

Under this rule, the short nominal interest rate behaves as follows, after eliminating $E_t \dot{c}$ using 8.61 and 8.63:

$$i = \left(\frac{\alpha\delta_1}{\alpha\delta_1 - 1}\right)(i^* - \pi^*) + \left(\frac{(\alpha - 1)\delta_1}{\alpha\delta_1 - 1}\right)\pi - \frac{1}{\alpha\delta_1 - 1}\bar{i}$$

The equations of motion of the system are

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\delta_1 - 1}{1 - \alpha\delta_1} \\ \beta\eta & \frac{\beta\gamma(1-\delta_1)}{1-\alpha\delta_1} \end{bmatrix} \begin{bmatrix} c \\ \pi \end{bmatrix} + \begin{bmatrix} \frac{1}{1-\alpha\delta_1} & 0 & -\frac{1}{1-\alpha\delta_1} & \frac{1}{1-\alpha\delta_1} \\ -\frac{\beta\gamma}{1-\alpha\delta_1} & \beta & \frac{\alpha\beta\gamma\delta_1}{1-\alpha\delta_1} & -\frac{\alpha\beta\gamma\delta_1}{1-\alpha\delta_1} \end{bmatrix} \begin{bmatrix} \bar{i} \\ f - \bar{y} \\ i^* \\ \pi^* \end{bmatrix} \tag{8.78}$$

The condition for saddlepoint stability is that $\frac{\delta_1 - 1}{1-\alpha\delta_1} > 0$. Clearly, since $0 \leq \alpha \leq 1$, this can be satisfied only if $\delta_1 > 1$. There now is a further constraint on the permissible values of δ_1, however: δ_1 cannot be too large. The condition for saddlepoint stability is

$$1 < \delta_1 < \alpha^{-1}$$

The steady state configuration of the system and steady-state response to shocks is the same under the *DGI* and the CPI inflation versions of the Taylor rule. Other aspects of the transitional response to shocks are, however, quantitatively under the two versions of the Taylor rule (compare 8.70 and 8.78).

8.3.2 A Monetary Growth Rule

For completeness we provide the state-space representation of the model when the growth rate of the nominal money stock is governed by the following Taylor or McCallum rule:

$$\mu = \bar{\mu} + \delta_1 \pi + \delta_2 y + \delta_3 c + \delta_4 \ell$$

The equations of motion are given by:

$$\begin{bmatrix} \dot{c} \\ \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \Psi \begin{bmatrix} \eta\lambda^{-1}k & -1 & -\lambda^{-1} \\ \beta\eta & \beta\gamma & \beta\gamma\lambda^{-1} \\ \delta_3\Psi^{-1} + \delta_2\eta & (\delta_1 - 1)\Psi^{-1} + \delta_2\gamma & \delta_4\Psi^{-1} + \delta_2\gamma\lambda^{-1} \end{bmatrix} \begin{bmatrix} c \\ \pi \\ \ell \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & \frac{\lambda^{-1}k}{1+\gamma\lambda^{-1}k} & 0 & -1 & 1 \\ 0 & \frac{\beta}{1+\gamma\lambda^{-1}k} & -\beta & 0 & 0 \\ 1 & \delta_2\Psi & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mu} \\ f \\ \bar{y} \\ i^* \\ \pi^* \end{bmatrix}$$

$$1 > \Psi = \frac{1}{1 + \gamma\lambda^{-1}k} > 0$$

The determinant of the state matrix, D, is given by:

$$D = \beta\eta\Psi^{-1}((1 - \delta_1)\lambda^{-1} + \delta_4)$$

The Trace T is given by

$$T = \Psi\left(\eta\lambda^{-1}k + \beta\gamma + \delta_2\gamma\lambda^{-1} + \delta_4\Psi^{-1}\right)$$

The characteristic equation of the state matrix is:

$$\begin{aligned} &\chi(\rho) \equiv \rho^3 - \left(\eta\lambda^{-1}k + \beta\gamma + \delta_2\gamma\lambda^{-1} + \delta_4\Psi^{-1}\right)\rho^2 \\ &+\Psi^{-1}\left[\beta\eta + (1 - \delta_1)\beta\gamma\lambda^{-1} + \delta_2\eta\lambda^{-1} + \delta_3\lambda^{-1} + \delta_4(\beta\gamma + \eta\lambda^{-1}k)\right]\rho \\ &-\eta\beta\Psi^{-2}\left[(1 - \delta_1)\lambda^{-1} + \delta_4\right] = 0 \end{aligned}$$

A necessary condition for this to have two roots with negative real parts and one positive roots is that $\chi(0) = -\eta\beta\Psi^{-2}\left[(1 - \delta_1)\lambda^{-1} + \delta_4\right] < 0$. This is the condition that the determinant D be positive. Another necessary condition is that $\chi''(0) = -2\left(\eta\lambda^{-1}k + \beta\gamma + \delta_2\gamma\lambda^{-1} + \delta_4\Psi^{-1}\right) > 0$. This is the condition that the trace, T, be negative. Note that neither condition involves the feedback coefficient on the real exchange rate, δ_3. It is clear that a saddlepoint configuration can be achieved with $\delta_2 = \delta_3 = 0$. In that case we require $\delta_4 < -\Psi(\beta\gamma + \eta\lambda^{-1}k) < 0$.

A saddlepoint configuration can then be achieved even with $\delta_1 = 0$.

8.3.3 The MCI: A Monetary Conditions Index

During the early 1990s, the Bank of Canada has used the concept of *monetary conditions* (the combination of the movement of the short nominal interest rate and the nominal exchange rate) as the operational target of policy. The description that follows is from Freedman [15]. Technically, the *monetary conditions index* or MCI, is defined as the combination of the changes in short-term interest rates (typically the 90-day commercial paper rate) and the multilateral exchange rate (the G-10 rate) from some arbitrary base period. The idea is to obtain as estimate of the effect of movements in these two variables on aggregate demand over time. For practical applications, the Bank of Canada has used the percentage point change in the interest rates and the percent change (appreciation!) in the nominal exchange rate, the latter weighted by a factor of 1/3. This is based on empirical work which purports to find that a 1 percentage point change in interest rates has about the same effect on aggregate demand over time as a three percent appreciation in the exchange rate. The measure is thus calculated as the equivalent of percentage point changes in interest rates. While the real MCI, based on the real interest rate and the real exchange rate, is the theoretically relevant measure, for practical purposes, the Bank of Canada has focussed on the nominal MCI.

The rationale for the MCI appears to be the following. Consider the open-economy IS curve 8.59. It implies that

$$c \equiv s + p^* - p = \frac{\gamma}{\eta}(i - \pi) + \frac{1}{\eta}(y - f)$$

So, holding constant y and f, $\frac{dc}{dr} = \frac{\gamma}{\eta}$. In the short run, with p, p^* and π predetermined, holding constant y and f, $\frac{ds}{di} = \frac{\gamma}{\eta}$. A 1% (100 basis points) increase in the short nominal (real) rate of interest has the same negative effect on aggregate demand as a $\frac{\gamma}{\eta}$ percent appreciation of the nominal (real) exchange rate (decline in c or s).

I must confess that I have never been able to make head or tails or the MCI as an indicator of monetary conditions. In the long run the real interest rate in a small open economy with perfect international capital mobility is set by the rest of the world. The long run real exchange rate depends on capacity output, the long-run fiscal stance and other exogenous demand components and the long-run real interest rate. In the long run, therefore, both the real exchange rate and the real interest rate indicate everything except monetary conditions.

In the short run, monetary policy does affect the nominal and real interest rate and exchange rate. This is true when monetary growth is the monetary instrument or when the short nominal interest rate is the monetary instrument. The problem is that out of steady state, the (real and nominal) interest rate and exchange rate also reflect the behaviour of all other non-monetary exogenous variables: past, current and anticipated future changes in $\bar{y}$, i^*, p^*, π^* and f all affect the current spot exchange rate and, unless the nominal short interest rate is pegged by policy, also the current interest rate. The MCI is about as accurate an index of monetary conditions as the change in the government budget deficit is an indicator of fiscal conditions, that it, it is highly unreliable.

Since the MCI is part of the policy discourse in open economies, however, it may be interesting to investigate how a simple policy rule that responds to the MCI would perform.

We can construct two versions, a nominal and a real one, of the MCI for our little model as follows. The first characterisation, MCI, adds the change in the short nominal rate from some base period (t_0) to the appreciation in the nominal exchange rate from that same base period, weighted by a coefficient $\sigma = \frac{\gamma}{\eta} > 0$. In the Canadian application, $\sigma = 1/3$. Note that the nominal interest rate is a pure number (an instantaneous rate of return, at a point in time), while the exchange rate component is the change, over some finite period, in the logarithm of the nominal exchange rate. The two are not dimensionally commensurate, and the weighting factor σ has to have dimensions '$(\frac{\text{instantaneous nominal interest rate at time } t}{\text{natural logarithm of the nominal exchange rate at time } t})$' to convert the logarithm of a relative price of two moneys into an instantaneous rate of return in one money.

$$MCI(t) \equiv i(t) - i(t_0) - \sigma[s(t) - s(t_0)] \tag{8.79}$$

Corresponding to this nominal MCI index is the real MCI index:

$$mci(t) \equiv r(t) - r(t_0) - \sigma[c(t) - c(t_0)] \tag{8.80}$$

When considering policy feedback from an MCI index, a nominal interest rate rule does not produce any new results beyond those obtainable from the general nominal interest rate rules or 8.66. Consider for instance the following nominal interest rate feedback rule:

$$i = \bar{i} - \theta mci(t) \tag{8.81}$$

Presumably, if the MCI index increases, monetary policy is loosened, so $\theta > 0$. Equation 8.81 can be written as

$$i = \frac{1}{1+\theta}\left(\bar{i} + \theta[r(t_0) - \sigma c(t_0)]\right)\bar{i} + \frac{\theta}{1+\theta}\pi + \frac{\theta\sigma}{1+\theta}c$$

As regards the dynamic properties of the economic system, this is equivalent to 8.65 with $\delta_1 = \frac{\theta}{1+\theta}$, $\delta_2 = 0$ and $\delta_3 = \sigma\delta_1$.

From 8.68 the system will have a saddlepoint configuration if and only if

$$(1-\delta_1)\beta[\eta + \gamma\sigma\delta_1(1-\eta)] < 0$$

, that is, if and only if

$$\frac{\beta}{1+\theta}[\eta + \left(\frac{\gamma\sigma\theta}{1+\theta}\right)(1-\eta)] < 0$$

With $\theta > 0$, this condition can never be satisfied. Since under 8.81 the trace of the state matrix is positive, we know that the system is completely unstable.

To investigate what happens when the growth rate of the nominal money stock is related to the MCI index, we consider the following nominal (8.82) and real (8.83) MCI rules

$$\mu(t) = \bar{\mu} + \delta MCI(t) \tag{8.82}$$

$$\mu(t) = \bar{\mu} + \delta mci(t) \tag{8.83}$$

If the MCI is interpreted as measuring the tightness of monetary conditions, presumably $\delta > 0$: tight monetary conditions suggest that money growth be increased. The nominal MCI rule has undesirable long-run properties (the growth rate of money is s function of the level of the nominal exchange rate, which need not be constant, even in the long run. We therefore focus on the real MCI rule 8.83.

The equations of motion under the real MCI rule are given by:

$$
\begin{bmatrix} \dot{c} \\ \dot{\pi} \\ \dot{\ell} \end{bmatrix} = \Psi \begin{bmatrix} \eta\lambda^{-1}k & -1 & -\lambda^{-1} \\ \beta\eta & \beta\gamma & \beta\gamma\lambda^{-1} \\ \delta\left(\lambda^{-1}k\eta - \sigma(1+\gamma\lambda^{-1}k)\right) & -\left(1+\delta+\lambda^{-1}k\gamma\right) & -\delta\lambda^{-1} \end{bmatrix} \begin{bmatrix} c \\ \pi \\ \ell \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & \frac{\lambda^{-1}k}{1+\gamma\lambda^{-1}k} & 0 & -1 & 1 \\ 0 & \frac{\beta}{1+\gamma\lambda^{-1}k} & -\beta & 0 & 0 \\ 1 & \frac{\delta\lambda^{-1}k}{1+\gamma\lambda^{-1}k} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mu} - \delta[r(t_0) - \sigma c(t_0)] \\ f \\ \bar{y} \\ i^* \\ \pi^* \end{bmatrix}
$$

The characteristic equation of the state matrix is

$$
\begin{aligned}
&\rho^3 - [\beta\gamma + \lambda^{-1}(\eta k - \delta)]\rho^2 \\
&+(1+\lambda^{-1}k\gamma)[\beta(\eta+\gamma\lambda^{-1}) - \lambda^{-1}\delta\sigma] - \eta\beta\lambda^{-1}(1+\lambda^{-1}k\gamma)^2 = 0
\end{aligned}
$$

For this to have two stable and one unstable characteristic root it is necessary and sufficient that

$$
\eta\beta\lambda^{-1}(1+\lambda^{-1}k\gamma)^2 > 0
$$

and that

$$
\beta\gamma + \lambda^{-1}(\eta k - \delta) < 0
$$

This can be achieved, provided δ, the degree of responsiveness of monetary growth to the MCI index is strong enough.

Chapter 9

Seigniorage and Solvency

9.1 Seigniorage

9.2 Solvency

The traditional financial performance criteria (government budget deficit limits, ceilings on domestic credit (expansion) by the banking system, ceilings on credit extended to the non-financial public sector, floors on international reserves) that are to be observed by countries under IMF programs are inherently myopic. Recent attempts to adjust, correct and extend these measures can all be interpreted as moves to take a longer-term view of fiscal, financial and monetary policy conditionality. This change of approach is motivated by a recognition of two facts. First, even if the private sector were myopic, longer-run fiscal-financial and monetary dynamics will inevitably work their way into future short-run economic performance. Today's long-run becomes tomorrows short run. Second, and probably even more importantly, private agents, far from being myopic, are forward-looking in wage and price behavior and in the positions they assume and the prices they are willing to pay in domestic and international financial market. This means that even short-run performance depends on current and past anticipations of medium-and long-term behaviour of policy instruments and variables exogenous to the economy in question. The long-run casts its shadow forward into the present through anticipating actions of rational, calculating private agents.

If a longer-run perspective is to be taken (even if we are only or mainly interested in short-term performance), the right place, indeed the only place, to start is the comprehensive balance sheet or present value budget constraint of the consolidated public sector. This will permit us to think systematically about three sets of issues central to the Fund's concerns. First, the

issue of government solvency or sovereign default. Second, the issue of financial crowding out of private saving and investment by goverment borrowing and third, the monetary (and hence the inflation) implications of alternative fiscal-financial-monetry programs.

9.2.1 Some Basic Accounting

We start in equation 9.1 from the basic single-period budget identity (sources and uses of funds) of the combined public sector or CPS, that is, consolidated (or combined) non-financial public sector (NFPS) and public sector financial institutions (PFI). The latter include the central bank and all other public sector financial agencies such as development banks, import-export banks and publicly owned commercial banks.

$$\begin{aligned} & C_t^g - T_t - E_t N_t^* - F_t + I_t^g - PRIV_t + i_t B_{t-1}^d + i_t^* E_t (B_{t-1}^* - R_{t-1}^*) \\ \equiv \; & B_t^d - B_{t-1}^d + E_t (B_t^* - B_{t-1}^*) + H_t - H_{t-1} - E_t (R_t^* - R_{t-1}^*) \end{aligned} \tag{9.1}$$

C_t^g is the nominal value of government consumption spending in period t.

T_t is the nominal value of taxes net of transfers and subsidies in period t. It includes, with a negative sign, social security health and retirment benefits and public sector pension benefits.

E_t is the nominal spot exchange rate (the domestic currency price of foreign exchange in period t.

N_t^* is the foreign currency value of foreign aid.

F_t is the nominal value of the gross cash flow from the public sector capital stock in period t.

I_t^g is the nominal value of gross domestic capital formation in the public sector in period t.

$PRIV_t$ is the nominal value of privatization proceeds in period t.

i_t is the nominal interest rate on domestic currency denominated public debt in period t.

B_{t-1}^d is the nominal face value of the net stock of domestic currency-denominated interest-bearing liabilities of the consolidated NFPS and PFI, including arrears, outstanding at the beginning of period t. This includes the net non-monetary financial liabilities of the central bank vis-a-vis the private sector and the foreign sector.

i_t^* is the nominal interest rate on foreign currency denominated public debt in period t.

B^*_{t-1} is the foreign currency face value of the net stock of foreign currency-denominated interest-bearing liabilities of the consolidated NFPS and PFI, excluding official foreign exchange reserves, but including arrears, outstanding at the beginning of period t. This includes the net non-monetary financial liabilities of the central bank that are denominated in foreign exchange.

R^*_{t-1} is the foreign currency value of the stock of official international reserves (denominated in foreign currency) at the beginning of period t.

H_{t-1} is the nominal stock of non-interest bearing base money or high-powered money outstanding at the beginning of period t.

We also define the following:

$$H_t \equiv CU_t + RR_t \tag{9.2}$$

$$P_t(K^g_t - K^g_{t-1}) \equiv I^g_t - DEP_t - \frac{P_t}{P^k_t} PRIV_t \tag{9.3}$$

$$DEP_t \equiv P_t \delta_t K^g_{t-1} \tag{9.4}$$

$$F_t \equiv P_t \rho_t K^g_{t-1} \tag{9.5}$$

CU_t is the nominal stock of domestic currency in the hands of the public at the end of period t.

RR_t is the nominal value of bank reserves held with the central bank at the end of period t.

P_t is the domestic GDP deflator in period t.

K^g_t is the public sector capital stock at the end of period t valued at current reproduction cost, that is, measured in physical units, which are assumed to be real GDP units. The nominal reproduction cost of public sector capital is therefore assumed to be the GDP deflator, although a capital production cost index distinct from the GDP deflator could be added without complications.

DEP_t is the nominal value of public sector capital consumption or depreciation in period t.

P^k_t is the domestic currency value of the price obtained for a unit of public sector capital privatized in period t.

δ_t is the proportional rate of physical depreciation of the public sector capital stock in period t.

ρ_t is the gross real cash (or financial) rate of return on public sector capital in period t.

The current or consumption account primary surplus (that is, the non-interest, non-investment, non-privatization) surplus of the consolidated public sector and central bank, S_t^c , is defined in equation 9.6

$$S_t^c \equiv T_t + E_t N_t^* - C_t^g \tag{9.6}$$

The conventionally defined primary (non-interest) surplus of the consolidated public sector and central bank, S_t , is defined in equation 9.7. Unlike S^c it includes capital formation and privatization.

$$\begin{aligned} S_t &\equiv S_t^c + F_t + PRIV_t - I^g{}_t && (9.7) \\ &\equiv S_t^c + F_t - [P_t(K_t^g - K_{t-1}^g) + DEP_t] + \left(\frac{P_t^k - P_t}{P_t^k}\right) PRIV_t \end{aligned}$$

Public sector gross dissaving by the consolidated public sector and central bank or the consumption account deficit of the consolidated public sector and central bank, D_t^c , is defined in equation 9.8.

$$D_t^c \equiv -S_t^c - (F_t - DEP_t) + i_t B_{t-1}^d + i_t^* E_t (B_{t-1}^* - R_{t-1}^*) \tag{9.8}$$

The conventionally defined financial deficit or borrowing requirement of the CPS, D_t , is defined in equation 9.9

$$\begin{aligned} D_t &\equiv D_t^c + I_t^g - DEP_t - PRIV_t && (9.9) \\ &\equiv D_t^c + P_t(K_t^g - K_{t-1}^g) - \left(\frac{P_t^k - P_t}{P_t^k}\right) PRIV_t \\ &\equiv -S_t + i_t B_{t-1}^d + i_t^* E_t (B_{t-1}^* - R_{t-1}^*) \end{aligned}$$

From equations 9.1 and 9.3 to 9.5 we obtain equation 9.10

$$\begin{aligned} & C_t^g - T_t - E_t N_t^* + [\frac{P_t - P_t^k}{P_t^k}] PRIV_t + i_t B_{t-1} + i_t^* E_t (B_{t-1}^* - R_{t-1}^*) - (F_t - DEP_t) \\ \equiv \; & -P_t(K_t^g - K_{t-1}^g) + B_t^d - B_{t-1}^d + E_t[B_t^* - R_t^* - (B_{t-1}^* - R_{t-1}^*)] + H_t - H_{t-1} \quad (9.10) \end{aligned}$$

We also define the following.

$$\sigma_t \equiv (H_t - H_{t-1})/(P_t Y_t) \tag{9.11}$$

σ_t is *seigniorage* as a fraction of GDP, that is the change in the nominal stock of base money divided by nominal GDP.

$$B_t \equiv B_t^d + E_t(B_t^* - R_t^*) \tag{9.12}$$

B_t is the nominal face value (measured in domestic currency) of the total net stock of non-monetary *financial* debt of the CPS at the end of period t.

$$\bar{B}_t \equiv B_t - P_t K_t^g \tag{9.13}$$

$\bar{B}_t$ is the nominal face value of the total net stock of non-monetary tangible liabilities of the government at the end of period t. It subtracts the public sector capital stock valued at current reproduction cost from the net stock of non-monetary financial liabilities .

In what follows, lower-case characters stand for the corresponding upper-case character as a fraction of GDP. Letting Y_t denote real GDP in period t, we therefore have:

$c_t^g \equiv C_t^g/(P_t Y_t)$; the public sector consumption-GDP ratio in period t.

$k_t^g \equiv K_t^g/Y_t$; the public sector capital stock at the end of period t as a fraction of period t GDP.

$\tau_t \equiv T_t/(P_t Y_t)$; taxes net of transfers as a fraction of GDP in period t.

$n_t^* \equiv E_t N_t^*/(P_t Y_t)$; foreign aid as a fraction of GDP in period t.

$dep_t \equiv DEP_t/(P_t Y_t)$; public sector capital depreciation as a fraction of GDP.

$f_t \equiv F_t/(P_t Y_t)$; public sector capital income as a fraction of GDP.

$priv_t \equiv PRIV_t/(P_t Y_t)$; privatization receipts as a fraction of GDP in period t.

$b_t^d \equiv B_t^d/(P_t Y_t)$; the ratio of the net stock of interest-bearing debt denominated in domestic currency at the end of period t to period t GDP.

$b_t^* \equiv E_t B_t^*/(P_t Y_t)$; the ratio of the net stock of interest-bearing debt denominated in foreign currency at the end of period t to period t GDP.

$\rho_t^* \equiv E_t R_t^*/(P_t Y_t)$; the ratio of the stock of foreign exchange reserves at the end of period to to period t GDP.

$s_t^c \equiv S_t^c/(P_t Y_t)$; the current or consumption account primary surplus as a fraction of GDP.

$s_t \equiv S_t/(P_t Y_t)$; the primary surplus as a fraction of GDP.

$d_t^c \equiv D_t^c/(P_tY_t)$; CPS gross dissaving or the CPS current account deficit, as a fraction of GDP.

$d_t \equiv D_t/(P_tY_t)$; the CPS financial deficit or borrowing requirement as a fraction of GDP.

$b_t \equiv B_t/(P_tY_t)$; the nominal value of the total net stock of non-monetary *financial* liabilities of the CPS at the end of period t as a fraction of period t *GDP*.

$\bar{b}_t \equiv \bar{B}_t/(P_tY_t)$; the nominal value of the total net stock of non-monetary liabilities of the CPS at the end of period t, .as a fraction of period t *GDP*.

We also define the following:

$\pi_t \equiv (P_t/P_{t-1}) - 1$; the rate of inflation in period t.

$g_t \equiv (Y_t/Y_{t-1}) - 1$; the rate of growth of real GDP in period t.

$\epsilon_t \equiv (E_t/E_{t-1}) - 1$; the rate of depreciation of the nominal exchange rate in period t.

$r_t \equiv [(1 + i_t)/(1 + \pi_t)] - 1$; the period t domestic real interest rate.

We now can rewrite the identity in equation 9.10 as equation 9.14 or 9.15 below.

$$b_t \equiv \frac{1}{(1+\pi_t)(1+g_t)} b_{t-1} + \frac{\epsilon_t}{(1+\pi_t)(1+g_t)} (b^*_{t-1} - \rho^*_{t-1}) + d_t - \sigma_t \quad (9.14)$$

$$b_t \equiv \left(\frac{1+r_t}{1+g_t}\right) b_{t-1} + \left(\frac{(1+\epsilon_t)(1+i^*_t) - (1+i_t)}{(1+\pi_t)(1+g_t)}\right) (b^*_{t-1} - \rho^*_{t-1}) - s_t - \sigma_t \quad (9.15)$$

The term $\left(\frac{(1+\epsilon_t)(1+i^*_t)-(1+i_t)}{(1+\pi_t)(1+g_t)}\right) (b^*_{t-1} - \rho^*_{t-1})$ corrects for possible deviations from uncovered international interest parity. This is necessary because in the term $\left(\frac{1+r_t}{1+g_t}\right) b_{t-1}$, all debt has the domestic real interest rate imputed to it.

Identities 9.14 and 9.15 follow the unhelpful but common practice of lumping together current transactions and capital transactions in the primary deficit, $-s_t$, and in the financial deficit, d_t. It would be conceptually cleaner to use the alternative representations of equations 9.14 and 9.15 given in equations 9.16 and 9.17 below. Note that in equation 9.16, (minus) net public sector capital income $-(f_t - dep_t)$ is grouped together with interest payments $[i_tB^p_{t-1} + i^*_tE_t(B^*_{t-1} - R^*_{t-1})]/(P_tY_t)$, while privatization and (the negative of) net public sector capital formation are treated as financing items on a par with explicit borrowing.

$$\begin{aligned}\bar{b}_t \;\equiv\;& \frac{1}{(1+\pi_t)(1+g_t)}\bar{b}_{t-1} + d_t^c \qquad (9.16)\\ &+ \left(\frac{P_t - P_t^k}{P_t^k}\right) priv_t - \frac{\pi_t}{(1+\pi_t)(1+g_t)}k_{t-1}^g + \frac{\epsilon_t}{(1+\pi_t)(1+g_t)}(b_{t-1}^* - \rho_{t-1}^*)\\ &-\sigma_t\end{aligned}$$

Where

$$d_t^c \equiv -s_t^c + \frac{i_t}{(1+\pi_t)(1+g_t)}b_{t-1} + \frac{i_t^*(1+\epsilon_t)}{(1+\pi_t)(1+g_t)}(b_{t-1}^* - \rho_{t-1}^*) - \left(\frac{\rho_t - \delta_t}{1+g_t}\right)k_{t-1}^g$$

and

$$-s_t^c \equiv c_t^g - \tau_t - n_t^*$$

$$\begin{aligned}\bar{b}_t \;\equiv\;& \left(\frac{1+r_t}{1+g_t}\right)\bar{b}_{t-1} - s_t^c \qquad (9.17)\\ &+ \left(\frac{P_t - P_t^k}{P_t^k}\right) priv_t + \left(\frac{r_t - (\rho_t - \delta_t)}{1+g_t}\right)k_{t-1} + \left(\frac{(1+\epsilon_t)(1+i_t^*) - (1+i_t)}{(1+\pi_t)(1+g_t)}\right)(b_{t-1}^* - \rho_{t-1}^*)\\ &-\sigma_t\end{aligned}$$

Note from equation 9.17 that three conceptually different valuations of public sector capital are relevant for the evolution of the public sector's stock of non-monetary debt. The first is the current reproduction cost of a unit of public sector capital, assumed to be equal to P_t. This enters the definition of net non-monetary liabilities as a fraction of GDP, $\bar{b}_t \equiv [B_t^d + E_t(B_t^* - R_t^*) - P_tK_t^g]/(P_tY_t)$. The second is the price obtained by the government for a unit of privatized public sector capital, P_t^k. Equations9.16 and 9.17 emphasize the obvious point that privatization only reduces the rate of increase in the government's net non-monetary liabilities if the privatization price exceeds the current reproduction cost of government capital [1] . The third is the "continuation value" of a unit of public sector capital in the public sector, V_t . If the public sector capital stock is expected to remain in the public sector forever, the continuation value in the public sector of a unit of public

[1] Presumably, with a rational private sector, P_t^k cannot exceed the present discounted value of the quasi-rents the capital is expected to earn after privatization.

sector capital is given by equation 9.18 (if the discount rates i_{t+j} are treated as non-stochastic).

$$V_t = \lim_{N\to\infty} E_t \sum_{k=0}^{N-1} \prod_{j=0}^{k} \left(\frac{1}{1+i_{t+j}} \right) P_{t+j}(\rho_{t+j} - \delta_{t+j}) \tag{9.18}$$

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Thus the value to the government of a unit of public sector capital that is expected always to remain in the public sector is the present discounted value of its expected future quasi-rents in the public sector. The continuation value of the public sector capital stock in the public sector does not show up directly (as V_t) in the budget constraints, but the stream of quasi-rents whose present discounted value it is, does appear.

9.2.2 Fiscal Sustainability Indicator No 1: The Public Debt-GDP Ratio

While in the never-never land of pure theory, unbounded debt-GDP ratios are not inconsistent with government solvency and sustainable policy, *de facto* debt-GDP ratios will of course have to remain bounded. Especially at the beginning of a stabilization program, a government trying to establish or regain credibility, may use its a (sequence of) declining public debt-GDP ratio(s) as a signal of its ability to maintain long-run solvency. Equations 9.14 or 9.15 describe what will happen to the debt-GDP ratio during the next period.

Equation 9.14 requires as data inputs the growth rate of nominal GDP $(1+\pi_t)(1+g_t)-1$, the initial total public debt-GDP ratio b_{t-1}, the proportional rate of depreciation of the nominal exchange rate ϵ_t, the initial net

[2] If the objective of the government's privatization policy were to maximize its net worth, V_t would, under risk-neutrality, be determined from the following recursion relation: (α) $V_t = \frac{P_t(\rho_t - \delta_t)}{1+i_t} + Max\ E_t[\frac{V_{t+1}\ ,\ P^k_{t+1}}{1+i_t}]$ If the public sector capital stock is expected to remain in the public sector forever, equation (α) implies equation (β) (if the discount rates i_{t+j} are treated as non-stochastic). (β) $V_t = \lim_{N\to\infty} E_t \sum_{k=0}^{N-1} \prod_{j=0}^{k} \left(\frac{1}{1+i_{t+j}}\right) P_{t+j}(\rho_{t+j} - \delta_{t+j}) + \lim_{N\to\infty} E_t \prod_{j=0}^{N-1} \left(\frac{1}{1+i_{t+j}}\right) V_{t+1+j}$ If speculative bubbles are ruled out, $\lim_{N\to\infty} E_t \prod_{j=0}^{N-1} \left(\frac{1}{1+i_{t+j}}\right) V_{t+1+j} = 0$ and equation (β) expresses the value to the government of a unit of public sector capital that is expected alway to remain in the public sector as the present discounted value of its expected future quasi-rents in the public sector.

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foreign debt-GDP ratio $b^*_{t-1} - \rho^*_{t-1}$, the financial surplus of the public sector d_t and seigniorage as a proportion of GDP σ_t.

Equation 9.15 requires as data inputs the growth rate of real GDP g_t, the domestic real interest rate r_t, the initial total public debt-GDP ratio b_{t-1}, the proportional rate of depreciation of the nominal exchange rate (ϵ_t), the initial net foreign debt-GDP ratio $b^*_{t-1} - \rho^*_{t-1}$, the growth rate of nominal GDP $(1+\pi_t)(1+g_t)-1$, the foreign nominal interest rate i^*_t, the domestic nominal interest rate i_t, the primary surplus of the public sector s_t and seigniorage as a proportion of GDP σ_t.

Pitfalls

The financial deficit-GDP ratio and the primary deficit-GDP ratios can be distorted (made to look smaller in the short-run at the expense of being made larger in the long run) by the following tricks:

(1) Speeding up privatization receipts, if the privatized assets would have yielded positive future net cash flow to the government had they remained in the public sector (that is, if they had a positive continuation value in the public sector).

(2) Cutting back government capital formation, if the present discounted value of the future net cash flow to the government from these investment projects would have exceeded the cost of the projects.

To avoid these problems, it may be preferable to look at the behavior (in the past and anticipated for the next few years) of the public sector non-monetary financial debt net of the public sector capital stock valued at current reproduction cost, $\bar{b}$, given in equations 9.16(15) and 9.17 rather than at the non-monetary financial debt b, given in equations 9.14 and 9.15.

9.2.3 Fiscal Sustainability Indicator No 2: Primary Gaps

For notational simplicity, we now define the augmented primary surplus-GDP ratio, $\tilde{s}$, as follows:

$$\tilde{s}_t \equiv s_t - \frac{[i^*_t(1+\epsilon_t) + \epsilon_t - i_t]}{(1+\pi_t)(1+g_t)}(b^*_{t-1} - \rho^*_{t-1}) + \sigma_t \tag{9.19}$$

The augmented primary surplus therefore adds to the conventional primary surplus both seigniorage (the resources appropriated by the government by printing non-interest-bearing base money) and the excess of the cost of borrowing by issuing domestic currency denominated debt over the cost of

borrowing by issuing foreign currency denominated debt times the net stock of foreign debt.

Given the initial value of the total non-monetary government debt-GDP ratio at the beginning of period t, b_{t-1}, the target value of the debt-GDP ratio $N \geq 1$ periods later, b_{t-1+N} , the projected future one-period real interest rates during the next N periods, r_{t+j}, $j = 0, ..., N$, and the projected growth rates of real GDP during the next N periods, g_{t+j}, $j = 0, ..., N-1$, the constant augmented primary surplus to GDP ratio, $\tilde{s}_R^N$, that will achieve the target is given by:

$$\tilde{s}_R^N(b_{t-1} - b_{t-1+N}) = \left[\sum_{k=0}^{N-1} \prod_{j=0}^{k} \left(\frac{1+g_{t+j}}{1+r_{t+j}}\right)\right]^{-1} \left[b_{t-1} - \prod_{j=0}^{N-1} \left(\frac{1+g_{t+j}}{1+r_{t+j}}\right) b_{t-1+N}\right] \tag{9.20}$$

We shall refer to $\tilde{s}_R^N$ as the *required* N*-period* (augmented) *primary surplus-GDP ratio.* With constant real N-period interest rates r^N and constant N-period growth rates of real GDP g^N, the required N-period primary surplus-GDP ratio simplifies to:

$$\tilde{s}_R^N(b_{t-1} - b_{t-1+N}) = \frac{\left(r_t^N - g_t^N\right)}{\left(1+g_t^N\right)\left[1 - \left(\frac{1+g_t^N}{1+r_t^N}\right)^N\right]} \left[b_{t-1} - \left(\frac{1+g_t^N}{1+r_t^N}\right)^N b_{t-1+N}\right] \tag{9.21}$$

If the target debt-GDP ratio is the same as the initial debt-GDP ratio, equation the required

N-period primary surplus-GDP ratio simplifies to

$$\tilde{s}_R^N(0) = \left[\sum_{k=0}^{N-1} \prod_{j=0}^{k} \left(\frac{1+g_{t+j}}{1+r_{t+j}}\right)\right]^{-1} \left[1 - \prod_{j=0}^{N-1} \left(\frac{1+g_{t+j}}{1+r_{t+j}}\right)\right] b_{t-1} \tag{9.22}$$

With a constant N-period real interest rate $r^N$and a constant N-period growth rate of real GDP g^N, the required N-period primary surplus-GDP ratio for this case becomes

$$\tilde{s}_R^N(0) = \frac{\left(r_t^N - g_t^N\right)}{1+g_t^N} b_{t-1} \tag{9.23}$$

When $N = 1$, equation 9.23 simplifies to

$$\tilde{s}_R^1(0) = \frac{(r_t - g_t)}{1 + g_t} b_{t-1} \tag{9.24}$$

We also define the *actual N-Period (augmented) primary surplus-GDP ratio*, $\tilde{s}_A^N$, to be that constant augmented primary surplus-GDP ratio whose present discounted value over the next N periods is the same as the present discounted value of the actually planned or expected primary surplus-GDP ratio over the next N periods, that is,

$$\tilde{s}_A^N(t) = \left[\sum_{k=0}^{N-1} \prod_{j=0}^{k} \left(\frac{1 + g_{t+j}}{1 + r_{t+j}} \right) \right]^{-1} \left[1 - \prod_{j=0}^{N-1} \left(\frac{1 + g_{t+j}}{1 + r_{t+j}} \right) \right] b_{t-1} \tag{9.25}$$

When the real interest rate and the real growth rate are constant, equation 9.25 simplifies to

9.26

$$\tilde{s}_A^N = \frac{\left(r_t^N - g_t^N\right)}{\left(1 + g_t^N\right) \left[1 - \left(\frac{1+g_t^N}{1+r_t^N} \right)^N \right]} \sum_{k=1}^{N} \left(\frac{1 + g_t^N}{1 + r_t^N} \right)^k \tilde{s}_{t-1+k} \tag{9.26}$$

The N-period *primary gap* (see Blanchard et. al. [1990]) in period t, GAP_t^N , is defined as the excess of the rquired N-period (augmented) primary surplus-GDP ratio, $\tilde{s}_R^N$, over the actual N-period (augmented) primary surplus -GDP ratio, $\tilde{s}_A^N$:

$$GAP_t^N \equiv \tilde{s}_R^N (b_{t-1} - b_{t-1+N}) - \tilde{s}_A^N \tag{9.27}$$

Given an initial debt-GDP ratio and a target debt-GDP ratio for N periods hence, the N-period primary gap can be calculated using forecasts for real interest rates, real growth rates and actual planned or expected augmented primary surplus-GDP ratios.

Note that primary gaps defined with reference to the public sector non-monetary financial debt net of the value of the public sector capital stock, $\bar{b}$, can be defined exactly analogously. The relevant primary surplus would be $\tilde{s}^c$,the *augmented* current or consumption account primary surplus (as a fraction of GDP), defined by

$$\tilde{s}_t^c \equiv s_t^c - \frac{[i_t^*(1+\epsilon_t)+\epsilon_t - i_t]}{(1+\pi_t)(1+g_t)}(b_{t-1}^* - \rho_{t-1}^*) + \sigma_t \tag{9.28}$$

Two particularly simple special cases of the N-period primary gap provide the second and third fiscal indicators.

The One-Period Primary Gap

When $N = 1$and the initial debt-GDP ratio is the same as the target debt-GDP ratio at the end of the period, the primary gap calculation simplifies to:

$$GAP_t^1 \equiv \tilde{s}_R^1(0) - \tilde{s}_A^1 = \left(\frac{r_t - g_t}{1+g_t}\right) b_{t-1} - \tilde{s}_t \tag{9.29}$$

Note that the calculation of 9.29 does not require any forecasts other than those going into the calculation of the current real interest rate and current growth rate of real GDP. GAP_t^1, the one-period primary gap in period t is the excess of the augmented primary surplus-GDP ratio that stabilizes this period′s debt-GDP ratio over the actual current augmented primary surplus-GDP ratio.

Apart from the problems arising from the treatment of public sector capital formation and privatization, the one-period primary gap may give a distorted picture of the amount of adjustment that would reasonably be required for two further reasons. The first is that the actual current primary surplus may be affected by transitory (e.g. cyclical) increases or reductions in public sector revenues and non-interest expenditures. A cyclical correction may therefore be in order. The second is that the current real interest rate and growth rate of real GDP may be unrepresentative of their respective long-run expected average values. This suggest need for the longer-run perspective adopted in the next section.

Long-Run Solvency

When the long-run interest rate is above the long-run growth rate of GDP, that is if $\lim_{N\to\infty} \prod_{j=0}^{N-1} \left(\frac{1+g_{t+j}}{1+r_{t+j}}\right) = 0$, the following government solvency constraint is assumed to hold for an economy without a finite terminal date

$$lim_{N\to\infty} \prod_{j=0}^{N-1} \left(\frac{1+g_{t+j}}{1+r_{t+j}}\right) b_{t-1+N} = 0 \tag{9.30}$$

When equation 9.30 holds, the present value budget constraint given in 9.31 applies to the government. This states that the current face value of the debt cannot exceed the present discounted value of future primary surpluses and seigniorage.

$$b_{t-1} = lim_{N\to\infty} \sum_{k=0}^{N-1} \prod_{j=0}^{k} \left(\frac{1+g_{t+j}}{1+r_{t+j}} \right) \tilde{s}_{t+j} \tag{9.31}$$

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We can define the *required permanent (augmented) primary surplus-GDP ratio,* $\tilde{s}_R^\infty$ as follows:

$$\tilde{s}_R^\infty = lim_{N\to\infty} \left[\sum_{k=0}^{N-1} \prod_{j=0}^{k} \left(\frac{1+g_{t+j}}{1+r_{t+j}} \right) \right]^{-1} b_{t-1} \tag{9.32}$$

When the real interest rate and the growth rate of real GDP are constant, equation 9.32 becomes

$$\tilde{s}_R^\infty = \left(\frac{r_t^\infty - g_t^\infty}{1+g_t^\infty} \right) b_{t-1} \tag{9.33}$$

The required permanent (augmented) primary surplus-GDP ratio is the constant (augmented) primary surplus GDP ratio that, if maintained indefinitely, would ensure government solvency. It is also the constant primary surplus-GDP ratio that will ensure that, ultimately, the debt-GDP ratio does not exceed any finite upper limit (including its current value b_{t-1}).

The Permanent Primary Gap

The *permanent primary gap*, GAP_t^∞, was proposed in Buiter [1983, 1985 and 1990a]. It measures the magnitude of the permanent correction required to be made to the actual current and future planned augmented primary surplus-GDP ratios in order to ensure government solvency. This measure was also proposed in Blanchard [1990] and in Blanchard et. al. [1990]. It is given by the excess of the required permanent primary surplus-GDP ratio over actual permanent primary surplus-GDP ratio:

[3] Note that, when the long-run interest rate exceeds the long-run growth rate, government solvency is consistent even with unbounded (and forever rising) debt-GDP ratios as long as, ultimately, the public debt grows at a rate less than the interest rate.

$$\begin{aligned} GAP_t^{\infty} &= \tilde{s}_R^{\infty} - \tilde{s}_A^{\infty} \qquad (9.34) \\ &= lim_{N\to\infty}\left[\sum_{k=0}^{N-1}\prod_{j=0}^{k}\left(\frac{1+g_{t+j}}{1+r_{t+j}}\right)\right]^{-1}\left[b_{t-1}-\sum_{k=0}^{N-1}\prod_{j=0}^{k}\left(\frac{1+g_{t+j}}{1+r_{t+j}}\right)\tilde{s}_{t+j}\right] \end{aligned}$$

When the real interest rate and the growth rate of real GDP are constant, becomes

$$GAP_t^{\infty} = \left(\frac{r_t^{\infty}-g_t^{\infty}}{1+g_t^{\infty}}\right)[b_{t-1}-\sum_{k=1}^{N}\left(\frac{1+g_t^{\infty}}{1+r_t^{\infty}}\right)^k \tilde{s}_{t-1+k}] \qquad (9.35)$$

The calculation of the permanent primary gap requires forecasts of the long-run real interest rate and the long-run real growth rate and of the future primary surpluses that would materialize under current spending and revenue-raising plans. The lazy man's (or myopic) alternative, measured by $MGAP_t^{\infty}$, substitutes the current augmented primary surplus-GDP ratio, $\tilde{s}_t$, for the actual permanent augmented primary surplus-GDP ratio, that is

$$MGAP_t^{\infty} = \left(\frac{r_t^{\infty}-g_t^{\infty}}{1+g_t^{\infty}}\right)b_{t-1} - \tilde{s}_t \qquad (9.36)$$

This is the same as the one-period gap, except for the substitution of the long real interest rate for the current real interest rate and the substitution of the long-run growth rate of real GDP for the current growth rate of real GDP.

9.2.4 Fiscal Sustainability Criterion No 3: Tthe Discounted Public Debt.

The government solvency constraint

$\lim_{N\to\infty}\prod_{j=0}^{N-1}\left(\frac{1+g_{t+j}}{1+r_{t+j}}\right)b_{t-1+N} \equiv \lim_{N\to\infty}\prod_{j=0}^{N-1}\left(\frac{1}{1+i_{t+j}}\right)B_{t-1+N} = 0$,

suggests that the behavior of the discounted public debt $PDV(b_t)$ (the present discounted value of the public debt discounted to a fixed initial date, t_0, say), can serve as a useful indicator of potential trouble. Note that the discounted debt rises if and only if the augmented primary surplus is negative. If the discounted debt has been rising significantly and looks like continuing to do so in the foreseeable future, good reasons must be given why this upward trend will eventually be reversed.

The measure is given by

$$PDV(b_t) \equiv \prod_{j=0}^{t-t_o} \left(\frac{1}{1+i_{t_o+j}} \right) B_t \tag{9.37}$$

9.2.5 Fiscal Sustainability Criterion No 4: The Long-run Inflation Rate Implied by the Fiscal-Financial Plans

This is a standard financial programming exercise. For a reference, see *e.g.* Buiter [1993], Anand and van Wijnbergen [1989] and Buiter [1990b].

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Chapter 10

Policy Under Uncertainty

10.1 Policy Ineffectiveness

10.2 Gradualism and Caution

Chapter 11

Private Consumption and Portfolio Choice

11.1 Overlapping Generations Models

11.1.1 The Allais-Samuelson Overlapping Generations Model

Households are born each period and live for $\pounds \geq 2$ periods. The life span, $\pounds$, is treated as certain. If follows that each household overlaps with $2\pounds - 2$ other generations during its lifetime. For instance, if $\pounds = 2$, you overlap with your parents while you are young and with your children while you are old (see Figure 11.1a). If $\pounds = 3$, you overlap with your grandparents when you are a child, with your parents when you are a child and during middle age, with your children during middle age and during old age, and with your grand children during old age (see Figure 11.1b).

Figure 11.1a,b here

In the simplest OLG model, households of the same generation are identical. This leaves two kinds of heterogeneity. First, among those currently alive, households will have different ages. What matters here is age in a forward-looking sense: the length of the remaining life span. Second, among all those currently alive, and future generations yet to be born. Redistribution between heterogeneous households will affect aggregate consumption (and, in principle, also aggregate labour supply) if individual household behaviour is government by the life-cycle hypothesis. Redistribution from the young to the old (from consumers with a long remaining life span to consumers with a short remaining life span) will raise current consumption by

the old by more than it reduces consumption by the young. Redistribution from future generations to current generations will of course increase aggregate consumption. While as regards direct physical transfers it is of course only possible to redistribute among overlapping generations (that is, among generations currently alive), the government can use a chained sequence of direct physical intergenerational transfers to redistribute life-time resources among non-overlapping generations.

11.1.2 A Simple Model

Households

We consider households that are not linked to earlier or to later generations, either though preferences (intergenerational caring) or though pooled budget constraints. When $\pounds = 2$, the lifetime utility function of a representative member of generation t is given by

$$\begin{array}{c} U_t = u(c_t^1, c_t^2) \\ c_t^1, c_t^2 \geq 0 \end{array} \tag{11.1}$$

This utility function is (at least!) three times continuously differentiable, defined over non-negative consumption sequences, strictly increasing in both its arguments and strictly quasi-concave. Most of the time we'll also assume it to be homothetic (that is, u is a monotone (increasing) function of a linear homogeneous function, $u = f\left(c_t^1 v(\frac{c_t^2}{c_t^1})\right)$, $f' > 0$) and to satisfy the Inada conditions,

$$\begin{array}{l} \lim_{c_t^1 \to 0} u_1(c_t^1, c_t^2) = \lim_{c_t^1 \to 0} f'\left(v - \frac{c^2}{c^1} v'\right) = \lim_{c_t^2 \to 0} u_2(c_t^1, c_t^2) = \lim_{c_t^1 \to 0} f'v' = \infty; \\ \lim_{c_t^1 \to \infty} u_1(c_t^1, c_t^2) \lim_{c_t^1 \to \infty} f'\left(v - \frac{c^2}{c^1} v'\right) = \lim_{c_t^2 \to \infty} u_2(c_t^1, c_t^2) = \lim_{c_t^1 \to \infty} f'v' = 0. \end{array}$$

Often the objective functional will be taken to be time-additive, with a constant pure rate of time preference, $\delta \geq -1$. The single-period utility function or felicity function for this case is denoted v.

$$U_t = v(c_t^1) + \frac{1}{1+\delta} v(c_t^2) \tag{11.2}$$

Four key features of the household's utility function are time preference, intertemporal subsitution, risk aversion and prudence.

A useful class of felicity functions is the HARA (hyperbolic absolute risk aversion) family. Its usefulness derives from the fact that is is the only class of concave felicity functions that generate a consumption function that is linear in wealth.

The HARA family is given by

$$\begin{array}{c} v(c) = \frac{\gamma}{1-\gamma}\left(\frac{\varsigma}{\gamma}c + \xi\right)^{1-\gamma} \\ \zeta > 0; \frac{\varsigma}{\gamma}c + \xi > 0; \gamma \neq 1; \xi > 0 \text{ if } \gamma = \infty \end{array} \quad 1 \tag{11.3}$$

$$\begin{array}{c} v(c) = \ln(\zeta c + \xi) \\ \gamma = 1; \zeta c + \xi > 0 \end{array} \tag{11.4}$$

$$\begin{array}{c} v(c) = -e^{-\frac{\zeta}{\xi}c} \\ \gamma = \infty \end{array} \tag{11.5}$$

Felicity functions should be increasing in consumption and strictly concave. For the HARA felicity function,

$$v'(c) = \zeta\left(\frac{\varsigma}{\gamma}c + \xi\right)^{-\gamma} > 0 \text{ iff } \frac{\varsigma}{\gamma}c + \xi > 0$$

and

$$v''(c) = -\zeta^2\left(\frac{\varsigma}{\gamma}c + \xi\right)^{-(1+\gamma)} < 0 \text{ if } \frac{\varsigma}{\gamma}c + \xi > 0$$

The HARA family includes the following frequently used special cases:

(1) The isoelastic (or constant elasticity of marginal felicity) function with a constant pure rate of time preference. It is the special case of 11.3 when $\frac{\xi}{\zeta} = 0$ (and, without loss of generality, $\zeta = 1$). For this felicity function, $v'(c) = c^{-\gamma} > 0$, and $v''(c) = -\gamma c^{-(1+\gamma)} < 0$ if and only if $\gamma > 0$. The two-period utility function then becomes

$$\begin{array}{l} U_t = \frac{1}{1-\gamma}\left(c_t^1\right)^{1-\gamma} + \left(\frac{1}{1+\delta}\right)\frac{1}{1-\gamma}\left(c_t^2\right)^{1-\gamma}, \quad \gamma \neq 1 \\ U_t = \ln c_t^1 + \frac{1}{1+\delta}\ln c_t^2 \quad , \gamma = 1 \end{array} \tag{11.6}$$

(2) The exponential function. This is the special case of 11.3 when $\gamma = \infty$, given in 11.5. For this felicity function, $v'(c) = \frac{\zeta}{\xi}e^{-\frac{\zeta}{\xi}c} > 0$, and $v''(c) = -\left(\frac{\zeta}{\xi}\right)^2 e^{-\frac{\zeta}{\xi}c} < 0$. The two-period utility function then becomes

$$U_t = -\left(e^{-\frac{\zeta}{\xi}c_t^1} + \frac{1}{1+\delta}e^{-\frac{\zeta}{\xi}c_t^2}\right)$$

(3) The quadratic function. This is the special case of 11.3 when $\gamma = -1$. The felicity function can be written as

$$v(c) = -\frac{1}{2}\zeta^2c^2 + \zeta\xi c - \frac{1}{2}\xi^2$$

Note that $v'(c) = -\zeta(\zeta c - \xi)$, which is positive only if $\zeta c - \xi < 0$, and that $v''(c) = -2\zeta < 0$.

Time preference

The (psychological) pure rate of time preference, denoted $\delta(c_t^2 = c_t^1)$, is defined as the marginal rate of substitution between c_t^2 and c_t^1 when $c_t^2 = c_t^1$, minus 1.

The marginal rate of substitution between c_t^2 and c_t^1,denoted $MRS(c_t^2, c_t^1)$, is

given by

$$MRS(c_t^2, c_t^1) \equiv -\left.\frac{dc_t^2}{dc_t^1}\right|_{dU_t=0} = \frac{u_1(c_t^1, c_t^2)}{u_2(c_t^1, c_t^2)}$$

Thus

$$\delta(c_t^2 = c_t^1) = \left.\frac{u_1(c_t^1, c_t^2)}{u_2(c_t^1, c_t^2)}\right|_{c_t^1 = c_t^2} - 1$$

Time preference is a measure of the consumer's impatience, that is, his terms on which he is willing to substitute current felicity for future felicity.

Intertemporal substitution

Intertemporal subsitution elasticities are a measure of the consumer's willingness to put up with a time-varying path of life-time consumption.

The intertemporal elasticity of substitution, $\epsilon(c_t^2, c_t^1)$, is defined as the elasticity of the ratio of consumption when old to consumption when young,

$\frac{c_t^2}{c_t^1}$, with respect to the marginal rate of substitution between c_t^2 and c_t^1. The marginal rate of substitution for the general time-additive and HARA cases (with a constant pure rate of time preference) is given by

$$MRS(c_t^2, c_t^1) = (1+\delta)\frac{v'(c_t^1)}{v'(c_t^2)} = (1+\delta)\left(\frac{\frac{\varsigma}{\gamma}c_t^2 + \xi}{\frac{\varsigma}{\gamma}c_t^1 + \xi}\right)^{\gamma}$$

For the isoelastic utility function,

$$\epsilon(c_t^2, c_t^1) \equiv \frac{d\ln\left(\frac{c_t^2}{c_t^1}\right)}{d\ln MRS(c_t^2, c_t^1)} = \gamma^{-1}$$

Note that for this utility function, the elasticity of marginal utility is constant and equal to $-\gamma$. For quasi-concavity, we require $\gamma \geq 0$ in this case.

A high value for the intertemporal subsitution elasticity means that households are quite willing to accept an uneven path of consumption over time, if the interterporal terms of trade make this attractive. The extreme case is an infinite intertemporal subsitution elasticity, that is, a linear utility function:

$$u(c_t^1, c_t^2) = c_t^1 + \frac{1}{1+\delta}c_t^2$$

A low value for the intertemporal substitution elasticity means that households attempt to smooth consumption over time even in the intertemporal terms of trade favour substitution towards earlier or later consumption. The extreme case is the zero intertemporal elasticit of substitution function. This reduces to Leontieff or fixed coefficient preferences preferences:

$$u(c_t^1, c_t^2) = \min\left\{c_t^1, \frac{1}{1+\delta}c_t^2\right\}$$

With Leontieff preferences, households are strict permanent income consumers: the deseire for a smooth (indeed a constant) consumption profile over time overrides and variations in the intertemporal terms of trade.

Logarithmic preferences represent the case of a unitary intertemporal substitution elasticity: income effects and substitution effects of changes in the intertemporal terms of trade will cancel each other out in this case.

Risk aversion.

It is rather awkward to discuss risk aversion in a model without risk or uncertainty.

Anticipating future developments, consider a consumer who obeys the expected utility hypothesis. The utility function is assumed to be time additive with a constant rate of time preference. The consumer therefore maximises

$$\mathfrak{E}_t u(c_t^1, c_t^2) = v(c_t^1) + \frac{1}{1+\delta}\mathfrak{E}_t v(c_t^2)$$

where $\mathfrak{E}_t$ is the mathematical expectation operator, conditional on the information available at the beginning of period t.

A consumer is risk averse if he only accepts bets that have actuarially favourable odds. This means that his felicity function is strictly concave, that is, $v''(c) < 0$.

The coefficient of absolute risk aversion, R^A, is defined as follows:

$$R^A \equiv -\frac{v''(c)}{v'(c)}$$

The coefficient of relative risk aversion, R^R, is defined as follows

$$R^R \equiv -\frac{v''(c)c}{v'(c)}$$

For the HARA felicity function, these risk aversion parameters are given by

$$R^A = \left(\frac{c}{\gamma} + \frac{\xi}{\zeta}\right)^{-1}$$

$$R^R = \left(\frac{1}{\gamma} + \frac{\xi}{c\zeta}\right)^{-1}$$

For the isolelastic felicity function this simplifies to

$$R^A = \frac{\gamma}{c}$$

$$R^R = \gamma$$

For the exponential felicity function, we have

$$R^A = \frac{\zeta}{\xi}$$

$$R^R = c\frac{\zeta}{\xi}$$

Impatience (time preference), that is, attitudes towards consumption smoothing over time (intertemporal substitititution) and attitudes towards consumption smoothing across states of nature (risk aversion) are conceptually quite distinct phsychological characteristics. The standard approach to choice under uncertainty, the expected utility hypothesis due to Von Neumann and Morgenstern, however, makes it impossible to treat intertemporal substitution and risk aversion independently: a consumer who is risk averse is also keen on smoothing consumption over time. In the isoelastic case, for instance, the coefficient of relative risk aversion, γ, is the reciprocal of the intertemporal substitution elasticity, γ^{-1}.

Households receive exogenous endowment income, that is, non-asset income, pay lump-sum taxes, have access to a household storage technology and can lend or borrow in a competitive loan market. Each household treats the market interest rate, the sequence of lump-sum taxes it faces and the sequence of endowments it receives as exogenous.

Let $e_t^1 \geq 0$ be the real endowment received by a young household in period t, $e_{t-1}^2 \geq 0$ the real endowment received by an old household in period t, r_{t+1} the one-period real interest rate in period t, τ_t^1 taxes paid while young by a member of generation t, and τ_{t-1}^2.taxes paid while old paid by a member of generation $t-1$.

Each household born in period t can lend an amount ℓ_t^1 while young (the *supply* of loans), at a competitive rate of interest r_{t+1}. In principle, they can also lend an amount ℓ_t^2 when old, at a competitive rate of interest r_{t+1}. When young, household also have access to a personal or household storage technology, which may be productive. For each unit of output allocated to storage in period t, the household gets back $1 + \pi_{t+1}$ units of output the next period, where $\pi \geq -1$. We treat the physical rate of return to storage, π, as constant and as the same for all households. The rate of

return to storage for a given household's technology could depend on the amount of real resources allocated to that storage technology, without this affecting any key results[2]. Even if the physical rate of return to storage in period t were a constant, π^i_{t+1}, say, for each household $i = 1, \ldots, N_t$ born in generation t, these returns could differ across households. With such intra-generational heterogeneity, there could be an operational capital rental market and an associated credit market, involving transactions among the young. The capital rental market would be a rental market for real resources to be used in the storage technology, that is, a rental market for capital services. In exchange for 1 additional unit of the real resource in period t, a competitive, utility maximizing household/producer, i, would be willing to offer a payment in period $t+1$ of at most $1+\pi^i_{t+1}$. This would be the highest real interest rate household i of generation t would be willing to pay for an additional one unit 'production loan' and the lowest interest rate it would have to receive in exchange for giving up one unit of output to be lent (as a production loan) to some other household of generation t.

In a competitive equilibrium all storage would occur through the household technology with the highest physical rate of return to storage, and the competitive rental rate would equal the physical rate of return to storage (the marginal product of capital) which would also equal the competitive real rate of interest. In what follows we assume for simplicity that the physical rates of return to storage are the same for all households of a generation and independent of the amount of storage.

The amount of real resources devoted to storage by a representative household of generation t is denoted σ^1_t; note that $\sigma^1_t \geq 0$ and that only the young can engage in storage.

The budget constraints of the generation t household are

$$e^1_t - c^1_t - \tau^1_t \geq \ell^1_t + \sigma^1_t \tag{11.7}$$

$$\begin{aligned} c^2_t + \tau^2_t + \ell^2_t \quad &\leq \quad e^2_t + (1 + r_{t+1}) && (11.8) \\ &\quad\ \ \ell^1_t + (1 + \pi_{t+1})\sigma^1_t && (11.9) \end{aligned}$$

$$\ell^2_t \geq 0 \tag{11.10}$$

[2] In principle, it could also depend on the amoung of resources allocated to storage by other households, but for simplicity such technological externalities are ignored.

equations 11.7 and 11.8 reflect the assumption that there are no intergenerational gifts or bequests. Equation 11.10 reflects the assumption that households cannot die in debt. With lifetime utility strictly increasing in consumption in both periods, all three weak equalities will hold as equalities. Specifically, households are legally banned from borrowing in the last period of their lives [3] and will have no incentive to be positive net lenders in the last period of their lives.

Equilibrium in the lending market occurs when there is no excess demand for loans at the prevailing interest rate. Total lending in period t is the sum of the lending by the young and the lending by the old. There are $N_t > 0$ members of generation t, and the generational growth rate is n_t, that is,

$$\begin{aligned} &N_{t+1} = (1 + n_{t+1})N_t \\ &n_{t+1} > -1 \end{aligned}$$

The government

All the government does is levy lump-sum taxes or pay lump-sum transfers to households. There is no exhaustive public spending and no issuance of government debt instruments. This implies

$$\tau_t^1 N_t + \tau_{t-1}^2 N_{t-1} \equiv 0 \tag{11.11}$$

Equilibrium

Equilibrium in the loan market is given by:

$$\begin{gathered} \ell_t^1 N_t + \ell_{t-1}^2 N_{t-1} = 0 \\ \text{for } 0 < \tfrac{1}{1+r_{t+1}} < \infty \end{gathered} \tag{11.12}$$

$$\begin{gathered} \ell_t^1 N_t + \ell_{t-1}^2 N_{t-1} \leq 0 \\ \text{for } \tfrac{1}{1+r_{t+1}} = 0 \end{gathered} \tag{11.13}$$

$$\begin{gathered} \ell_t^1 N_t + \ell_{t-1}^2 N_{t-1} \geq 0 \\ \text{for } r_{t+1} = -1 \end{gathered} \tag{11.14}$$

[3] If households are not anonymous (and if a person's age is known), no lender would ever lend to a borrower who is in the last period of his life.

Corner solutions matter here, so we cannot avoid 11.13 and 11.14. Equation 11.13 says that if the price of period t consumption in terms of period $t+1$ consumption, $1+r_{t+1}$, is infinite (the price of period $t+1$ consumption in terms of period t consumption, $\frac{1}{1+r_{t+1}}$, is zero), there cannot be excess supply of loans. Equation 11.14 says that if the price of period t consumption in terms of period $t+1$ consumption is zero (the price of period $t+1$ consumption in terms of period t consumption is infinite), there cannot be excess demand for loans.

From 11.10, holding with equality, it follows that as long as the price of period t consumption in terms of period $t+1$ consumption is non-zero, equilibrium lending by the young is zero. Obviously when the price of period t consumption in terms of period $t+1$ consumption is zero ($r_{t+1} = -1$) the young will certainly not wish to lend. It follows that in equilibrium,

$$\ell_t^1 N_t = 0 \tag{11.15}$$

Thus in equilibrium, the young never do any positive lending (positive financial saving). The reason is obvious. Since all the young are the same, they could only lend to the old. The old will not be around next period when they would have to repay their loans with interest. The young might want to borrow from the old, but obviously, the old will not want to lend to them. The only competitive equilibrium is therefore one with zero financial saving:

$$\ell_t^1 = \ell_t^2 = 0 \text{ for all } t$$

What this implies for equilibrium interest rates depends on the utility functions, the endowment sequence, the tax sequence and the rate of return to productive storage. We consider a number of examples.

(1) No taxes; a perishable endowment during one's youth only; an isoelastic utility function.

With utility functions that satisfy the Inada conditions, no taxes ($\tau_t^1 = \tau_t^2 = 0$), endowments only during one's youth ($e_t^1 = 1$, $e_t^2 = 0$) and a perishable commodity ($\pi_{t+1} = -1$), the price of period t consumption in terms of period $t+1$ consumption, $1 + r_{t+1}$, would have to equal zero (the price of period $t+1$ consumption in terms of period t consumption $\frac{1}{1+r_{t+1}}$ would have to be infinite), for young households to desist from trying to shift some consumption towards the second period of their lives by saving and using the financial market.

The loan market is inactive, as it is in all 2-period OLG models without any intra-generational heterogeneity. The competitive equilibrium allocations are given by:

$$\ell_t^1 = \ell_t^2 = 0$$

$$c_t^1 = 1$$

$$c_t^2 = 0$$

It is clear that the familiar *interior* intertemporal first-order condition,

$$\left(c_t^1\right)^{-\gamma} = \frac{1 + r_{t+1}}{1 + \delta} \left(c_t^2\right)^{-\gamma} \tag{11.16}$$

can characterize a competitive equilibrium (which has $c_t^2 = 0$), only if $r_{t+1} = -1$. Consider the case where the indifference curves in $c^1 - c^2$ space are strictly convex to the origin ($\gamma > 0$).

With $c_t^2 = 0$ and $c_t^1 = 1$ in any competitive equilibrium, we would not necessarily (for a general utility function), expect a tangency solution for consumption, with an indifference curve tangent to the budget constraint. However, with the isoelastic utility function, this will be the case.

The slope of the indifference curves is

$$\left.\frac{dc_t^2}{dc_t^1}\right|_{U_t = \overline{U_t}} = -(1 + \delta) \left(\frac{c_t^2}{c_t^1}\right)^{\gamma}$$

which is zero whenever $c_t^2 = 0$. So there is indeed a rather degenerate tangency equilibrium with $r_{t+1} = -1$ for all t.

Efficiency. We distinguish two notions of efficiency: *dynamic efficiency* and *Pareto-efficiency.*

The decentralised competitive equilibrium of this simple example is not Pareto efficient, although it is dynamically efficient.

Dynamic efficiency.

Definition 1 *A feasible allocation is dynamically efficient, if there does not exist another feasible allocation that has more aggregate consumption at one date without having lower aggregate consumption at some other date.*

In the context of our model, aggregate consumption at time t is defined as $C_t \equiv c_t^1 N_t + c_{t-1}^2 N_{t-1}$. A feasible consumption allocation $\{\bar{c}_t^1, \bar{c}_t^2; -\infty < t < +\infty\}$ is dynamically inefficient if and only if there exists another feasible consumption allocation $\{\bar{\bar{c}}_t^1, \bar{\bar{c}}_t^2; -\infty < t < +\infty\}$,such that, $\overline{C}_{t+i} \equiv \bar{c}_{t+i}^1 N_{t+i} + \bar{c}_{t-1+i}^2 N_{t-1+i} < \overline{\overline{C}}_{t+i} \equiv \bar{\bar{c}}_{t+i}^1 N_{t+i} + \bar{\bar{c}}_{t-1+i}^2 N_{t-1+i}$ for some i and
$\overline{C}_{t+i} \leq \overline{\overline{C}}_{t+i}$ for all i.

The competitive equilibrium of our example is dynamically efficient. This is obvious, since the entire perishable endowment each period, N_t, is consumed without physical waste:

$$C_t \equiv c_t^1 N_t + c_{t-1}^2 N_{t-1} = c_t^1 N_t = N_t$$

Pareto efficiency

Definition 2 *A feasible allocation is Pareto efficient if there does not exist another feasible allocation in which at least one consumer is better off and no consumer is worse off.*

Note that if an allocation is dynamically inefficient, it is also Pareto-inefficient.

In the context of our simple model, this means that a feasible consumption allocation $\{\bar{c}_t^1, \bar{c}_t^2; -\infty < t < +\infty\}$ is Pareto inefficient if and only if there exists another feasible consumption allocation $\{\bar{\bar{c}}_t^1, \bar{\bar{c}}_t^2; -\infty < t < +\infty\}$,such that, $\overline{U}_{t+i} \equiv u(\bar{c}_{t+i}^1, \bar{c}_{t+i}^2) < \overline{\overline{U}}_{t+i} \equiv u(\bar{\bar{c}}_{t+i}^1, \bar{\bar{c}}_{t+i}^2)$ for some i and
$\overline{U}_{t+i} \leq \overline{\overline{U}}_{t+i}$ for all i.

It is clear that the competitive equilibrium allocation of our simple model is Pareto-inefficient. While none of the endowment gets wasted in an engineering sense (the equilibrium is dynamically efficient), the allocation of consumption across the old and the young alive at any one time and the allocation of consumption across the lifetime of every household in every generation is pretty woeful: the young consume the entire endowment; the old consume nothing. There is no scope for consumption smoothing over the life cycle and across generations.

The the competitive equilibrium allocation as a benchmark. Let the social planner, starting in period t, take a small amount, $\epsilon > 0$, of the endowment

from each member of generation t and give it to the members of generation t. Let the resources be distributed equally among all the old, that is each one gets $\epsilon(1+n_t)$. Assume this scheme is continued when generation $t+1$ is born and forever after. Instead of a life-time consumption sequence $\{1,0\}$, each generation starting in period t, will have a life-time consumption sequence $\{1-\epsilon, \epsilon(1+n_{t+1}\}$. Generation $t-1$, who were old in period t when this intergenerational redistribution scheme was introduced, will have the life-time consumption sequence $\{1, \epsilon(1+n_t)\}$ rather than $\{1,0\}$ without the scheme. Generation $t-1$ is obviously better of with the intergenerational redistribution scheme. With strictly quasi-concave utility functions (indifference curves that are strictly convex to the origin), there always exists a small enough value of ϵ that will make all generations t and beyond also better off. This is most easily seen when $n_t = 0$.

The Pareto inefficiency can be explained in a number of ways. The key reason is that there is an infinite number of generations following any generation. If the total number of generations were finite, say the world is known to come to an end after $t = t^F$,the competitive equilibrium would again be Pareto efficient. If would not be possible to tax the last generation, born in period t^F while young and compensate them when old (in period t^F+1 by taxing the generation that is young in period t^F+1, since there will be no period t^F+1 and no generation t^F+1. For chain letters to work, the population has to be infinite. While this example of Pareto inefficiency is cute, it is not economically relevant, since it crucially relies on a counterfactual assumption (there always is an infinite number of generations following any given generation). Whenever 'infinity' (of the number of consumers, the number of commodities, the number of time periods, the number of trading intervals etc.) is crucial to a proposition in economics (as opposed to being a convenient analytical simplification for results that can be established with more hard work for 'large but finite', the proposition is without economic content.

In our simple example, the equilibrium real interest rate is -1. This means that the price of period $t+1$ consumption in terms of period t consumption is infinite. At any time, t, the value (that is, the present discounted value), of the economy-wide endowment sequence,

$$V_t \equiv e_t^1 N_t + e_{t-1}^2 N_{t-1} + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \left(\frac{1}{1+r_{t+j}} \right) \left(e_{t+i}^1 N_{t+i} + e_{t-1+i}^2 N_{t-1+i} \right) \tag{11.17}$$

is unbounded, or not defined.

We appear to have an economy without proper scarcity, since the value

of the endowment sequence is infinite. It is not surprising that funny things happen in such an economy.

(2) No taxes; an endowment during one's youth only; productive storage; an isoelastic utility function.

With utility functions that satisfy the Inada conditions, no taxes ($\tau_t^1 = \tau_t^2 = 0$), endowments only during one's youth ($e_t^1 = 1$, $e_t^2 = 0$) and productive storage, ($\pi_{t+1} > -1$), the equilibrium changes in quite interesting ways.

There still can be no lending in equilibrium between the young and the old. When the endowment is not completely perishable, however, there is more scope for consumption-smoothing, even in a competitive economy without government, through storage.

The competitive real interest rate will again equal the physical rate of return to storage. If it did not, say $r_{t+1} < \pi_{t+1}$, a young household in period t would want to borrow from the other young households at a rate that would earn a pure profit $\pi_{t+1} - r_{t+1}$ per unit of the endowment borrowed.

Let $K_{t+1} \geq 0$ be the stock of capital in existence at the beginning of period $t+1$ (the end of period t). This is just the endowment carried forward for use in the household storage technology by the young of generation t, that is,

$$K_{t+1} = N_t \sigma_t^1 = N_t(1 - c_t^1) \tag{11.18}$$

Note that this implies that

$$N_{t-1} c_{t-1}^2 = (1 + \pi_t) K_t \tag{11.19}$$

Letting $k_t \equiv \frac{K_t}{N_t}$, we can write this in intensive form as follows,

$$(1 + n_{t+1}) k_{t+1} = 1 - c_t^1 \tag{11.20}$$

and

$$c_{t-1}^2 = (1 + n_t)(1 + \pi_t) k_t \tag{11.21}$$

Competitive equilibrium is characterised by:

$$1 + r_{t+1} = 1 + \pi_{t+1} > 0 \tag{11.22}$$

$$\left(c_t^1\right)^{-\gamma} = \frac{1+\pi_{t+1}}{1+\delta}\left(c_t^2\right)^{-\gamma} \tag{11.23}$$

$$c_t^1 + \frac{1}{1+\pi_{t+1}}c_t^2 = 1 \tag{11.24}$$

The interesting feature of this economy is that if $\pi_{t+1} \leq n_{t+1}$ for all t, a competitive equilibrium will be not only Pareto-inefficient, but also dynamically inefficient.

The easiest way to see this is to consider the same competitive economy, but with balanced-budget redistribution between the generations (an ideal, non-distortionary unfunded or pay-as-you-go social security retirement scheme, outlined in 11.11, with τ_t^1 constant for simplicity.

With the addition of lump-sum intergenerational redistribution, the equilibrium becomes:

$$(1+n_{t+1})k_{t+1} = 1 - \tau_t^1 - c_t^1 \tag{11.25}$$

and

$$c_{t-1}^2 = (1+n_t)(1+\pi_t)k_t + \tau_t^1(1+n_t) \tag{11.26}$$

The competitive equilibrium is characterised by:

$$1 + r_{t+1} = 1 + \pi_{t+1} > 0 \tag{11.27}$$

$$\left(c_t^1\right)^{-\gamma} = \frac{1+\pi_{t+1}}{1+\delta}\left(c_t^2\right)^{-\gamma} \tag{11.28}$$

$$c_t^1 + \frac{1}{1+\pi_{t+1}}c_t^2 = 1 - \tau_t^1 + \left(\frac{1+n_{t+1}}{1+\pi_{t+1}}\right)\tau_{t+1}^1 \tag{11.29}$$

The interesting feature of this economy is that if $\pi_{t+1} \leq n_{t+1}$ for all t, a competitive equilibrium with zero taxes will be not only Pareto-inefficient, but also dynamically inefficient.

The intuition is simple. Households wish to smooth consumption of the life cycle. They can only do so using the storage technology. This means that the economy will carry a positive capital stock at each point in time (the inventory of commodities used in the household storage technology). Consider the simple case there the economy is stationary: $\pi_t = \pi; n_t = n$. The equilibrium is stationary and has a constant, positive stock of inventories:

$$(1+n)k_{t+1} = 1 - \tau_t^1 - c_t^1 \tag{11.30}$$

$$c_{t-1}^2 = (1+n)(1+\pi)k_t + \tau_t^1(1+n) \tag{11.31}$$

$$1 + r = 1 + \pi > 0 \tag{11.32}$$

$$\left(c_t^1\right)^{-\gamma} = \frac{1+\pi}{1+\delta}\left(c_t^2\right)^{-\gamma} \tag{11.33}$$

$$c_t^1 + \frac{1}{1+\pi}c_t^2 = 1 - \tau_t^1 + \left(\frac{1+n}{1+\pi}\right)\tau_{t+1}^1 \tag{11.34}$$

Equivalently, the social planner is constrained only by the physical resource constraint sequence, non-negativity of consumption and capital stocks and the initial capital stock. Let the initial period be t_0.:

$$c_t^1, c_{t-1}^2, k_t \geq 0; t \geq t_0$$

$$k_{t_0} = \overline{k}_{t_0}$$

$$c_t^1 N_t + c_{t-1}^2 N_{t-1} + K_{t+1} - K_t = e_t^1 N_t + e_{t-1}^2 N_{t-1} + \pi_t K_t = N_t + \pi_t K_t$$

or

$$c_t^1 + \frac{c_{t-1}^2}{1+n_t} = 1 + (1+\pi_t)k_t - (1+n_{t+1})k_{t+1}$$

Assume that, in period t_0, the two generations alive at the time are allowed to consume what they would have consumed in the competitive equilibrium, *plus* the entire capital stock that would have been accumulated under the competitive scheme, that is, $\widetilde{K}_{t_0+1}$. Thereafter, no storage takes place, that is, $k_t = 0$, $t > t_0$.For concreteness, we assume that it is the young in period t_0 who consume the capital stock that would have been accumulated in the competitive case (in addition to what they would have consumed in the competitive case), while the old in period t_0 consume what they would have consumed in the competitive equilibrium.

The resource constraint, or intergenerational consumption possibility set in period t_0 is therefore given by:

$$c^1_{t_0} + \frac{c^2_{t_0-1}}{1+n_{t_0}} = 1 + (1+\pi_{t_0})\overline{k}_{t_0}$$

The old in period t_0 consume

$$\frac{c^2_{t_0-1}}{1+n_{t_0}} = (1+\pi_{t_0})\overline{k}_{t_0}$$

The young in period t_0 consume their entire endowment.

$$c^1_{t_0} = 1$$

The resource constraint or intergenerational consumption possibility set in period $t > t_0$ is given by

$$c^1_t + \frac{c^2_{t-1}}{1+n_t} = 1$$

Consider the special case where the economy is stationary $(n_t = n; \pi_t = \pi)$

In this case the life-time consumption possibilities of each generation except for generations $t_0 - 1$ and t_0 are the same as the intergenerational consumption possibility set:

$$c^1_t + \frac{c^2_t}{1+n} = 1; \quad t > t_0$$

This contrasts with the stationary competitive life-time consumption possibilities of the successive generations:

$$c^1_t + \frac{c^2_t}{1+\pi} = 1; t \geq t_0$$

With $\dot{\pi} \leq n$ the life-time consumption possibility set of all generations beyond t_0 (weakly) under the social planner (weakly) contains the competitive life-time consumption possibility set. as shown in Figure 11.2.

Figure 11.2 here

All generations born in period t_0+1 or later are therefore at least as well off as in the competitive equilibrium. If $n = \pi$, the planner could simply replicate the competitive equilibrium consumption allocations for all generations born in $t_0 + 1$ or later, allow generation t_0 to consume his entire endowment while young, and give them their competitive equilibrium consumption allocation when old. Compared with the competitive equilibrium, the social planner can allow generation t_0 to have more consumption during its first period and the same level of consumption during its second period. All other generations have identical consumption profiles. The competitive equilibrium is both Pareto inefficient and dynamically inefficient.

11.1.3 An OLG economy with a perishable consumer good and pet rocks

We now change the model by assuming that while consumer goods are perishable ($\pi_t = -1$) there is a non-consumable durable commodity. The commodity in question is without intrinsic value, either as a consumption good (it does not enter the utility function) or as a capital good (it is not an input into any kind of household storage technology that produces a return in the form of intrinsically useful commodities). We can think of this durable good as 'pet rocks'[4].

The stock of pet rocks is positive and its quantity at the initial date, $t = t_0$, is exogenously given. The aggregate stock of pet rocks at the beginning of period t is denoted M_t, the price of period t pet rocks in terms of the period t consumable good is P_t^M. Pet rocks can have a non-zero exogenous 'own rate of return', that is, a rate of return in terms of pet rocks. A unit of pet rocks held (stored) from period t to period $t+1$ grows into $1 + i_{t+1}^M$ units of pet rocks.

Pet rocks are an asset of their owner, and a liability of no one. In the initial period, t_0, the stock of pet rocks, M_{t_0} is owned by the old (those born in period $t_0 - 1$). A representative young household's demand for pet rocks at time t, is denoted $M_{t,t+1}^d$. The household budget constraints are:

[4]This would not work in California, where the natives actually like their pet rocks.

$$1 - c_t^1 = P_t^M M_{t,t+1}^d \tag{11.35}$$

$$c_t^2 = (1 + i_{t+1}^M) P_{t+1}^M M_{t,t+1}^d \tag{11.36}$$

In equilibrium, the demand for pet rocks equals the exogenous stock:

$$N_t M_{t,t+1}^d = M_{t+1}$$

$$M_{t+1} = (1 + i_{t+1}^M) M_t$$

Per capita pet rocks are denoted $m_t \equiv \frac{M_t}{N_t}$

The non-pet-rocky equilibrium.

Note first that there always exists an equilibrium in which the price of pet rocks is zero each period. This is the familiar Pareto-inefficient equilibrium in which each generation consumes the perishable endowment while young and does not consume anything while old

$$c_t^1 = 1$$

$$c_t^2 = 0$$

$$P_t^M = 0$$

In some sense, the zero price of pet rocks equilibrium is the only fundamental equilibrium. Pet rocks are without intrinsic value as consumption goods, capital goods or intermediate goods. Any positive valuation of pet rocks is a 'bubble'. A pet rock has value today only because it is expected to have value tomorrow, etc. ad infinitum. Some of these 'bubbles' are beneficial. As we shall see, stationary positive price of pet rocks equilibria are Pareto efficient.

A multitude of pet-rocky equilibria.

Equilibria with a non-zero price of pet rocks are characterised by:

$$\left(c_t^1\right)^{-\gamma} = \frac{\frac{P_{t+1}^M}{P_t^M}(1+i_{t+1}^M)}{1+\delta}\left(c_t^2\right)^{-\gamma} \tag{11.37}$$

$$c_t^1 + \left(\frac{1}{1+i_{t+1}^M}\right)\frac{P_t^M}{P_{t+1}^M}c_t^2 = 1 \tag{11.38}$$

$$c_t^1 + \frac{c_{t-1}^2}{1+n_t} = 1 \tag{11.39}$$

$$P_t^M = \left(1-c_t^1\right)\frac{1}{m_{t+1}}\left(\frac{1}{1+n_{t+1}}\right) \tag{11.40}$$

$$m_{t+1} = \left(\frac{1+i_{t+1}^M}{1+n_{t+1}}\right)m_t \tag{11.41}$$

Consider the case where the population growth rate is constant, $n_t = n$.First note that the first three equations, 11.37, 11.38 and [**?**] can be solved for the three 'real' variables, c_t^1, c_t^2 and the inflation factor for pet rocks, $\frac{P_{t+1}^M}{P_t^M}$. Consider the case where the population growth rate is constant, $n_t = n$.

Stationary equilibria

There exists a stationary solution for the real rate of return on pet rocks, $1 + r_{t+1}^M \equiv \frac{P_{t+1}^M}{P_t^M}(1+i_{t+1}^M)$ and life-cycle consumption profiles, given by

$$\left(c^1\right)^{-\gamma} = \frac{(1+r^M)}{1+\delta}\left(c^2\right)^{-\gamma} \tag{11.42}$$

$$c^1 + \frac{c^2}{1+r^M} = 1 \tag{11.43}$$

$$c^1 + \frac{c^2}{1+n} = 1 \tag{11.44}$$

While the real rate of return on pet rocks will be constant, the inflation rate of pet rocks need not be. From 11.47, it is clear that the inflation rate

of pet rocks will be constant only if the exogenous own rate of return on pet rocks, i^M, is constant.

$$P_t^M = \left(1 - c^1\right) \frac{1}{m_{t+1}} \left(\frac{1}{1+n}\right) \tag{11.45}$$

$$m_{t+1} = \left(\frac{1 + i_{t+1}^M}{1+n}\right) m_t \tag{11.46}$$

$$\frac{P_{t+1}^M}{P_t^M} = \frac{1 + r^M}{1 + i_{t+1}^M} = \frac{1+n}{1 + i_{t+1}^M} \tag{11.47}$$

From equations11.43 and 11.44 is follows that the real rate of return on pet rocks equals the rate of population growth. With the rate of return to private saving equal to the 'biological' rate of return and with perishable commodities, the stationary equilibrium with a positive price of pet rocks is therefore Pareto-efficient. To determine the price sequence of pet rocks rather than just their inflation rates, we must know the entire sequence of pet rock stocks.

Non-stationary equilibria

We continue to assume that the population growth rate is constant, $n_t = n$. For simplicity we also assume that the own rate of return on pet rocks is constant and for even greater notational simplicity, we assume it equals zero: $i_t^M = 0$.

In addition to the stationary equilibrium, there will in general exist infinitely many non-stationary equilibria. Note that with $i_t^M = 0$, the stock of pet rocks is constant and the growth factor of m,the per capita stock of pet rocks, is the reciprocal of the growth factor of the population. The equilibrium conditions for $t \geq t_0$ are

$$\left(c_t^1\right)^{-\gamma} = \frac{\frac{P_{t+1}^M}{P_t^M}}{1+\delta} \left(c_t^2\right)^{-\gamma} \tag{11.48}$$

$$c_t^1 + \frac{P_t^M}{P_{t+1}^M} c_t^2 = 1 \tag{11.49}$$

$$c_t^1 + \frac{c_{t-1}^2}{1+n} = 1 \tag{11.50}$$

$$P_t^M = \left(1 - c_t^1\right) \frac{1}{m_{t+1}} \left(\frac{1}{1+n}\right) \tag{11.51}$$

$$m_{t+1} = \left(\frac{1}{1+n}\right) m_t \tag{11.52}$$

This implies the following first-order difference equation for the price of pet rocks:

$$P_{t+1}^M = (1+\delta)^{\frac{1}{1-\gamma}} P_t^M \left(\frac{1}{P_t^M m_{t+1}} - 1\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{if } \gamma \neq 1 \tag{11.53}$$

$$P_t^M = \frac{1}{m_{t+1}} \text{ if } \gamma = 1$$

Consider the special case where $n = 0$ and m is constant. Both the non-logarithmic case, given in 11.53, and the logarithmic case have the constant solution $P_t^M = \frac{1}{m}$ and the non-pet-rocky solution, $P_t^M = 0$. The non-logarithmic case also has a continuum of non-constant solutions.

The price of pet rocks, an asset price determined in an efficient, forward-looking pet rock market, is a non-predetermined state variable. There is no obvious boundary condition motivated by economic theory or empirics, that permits us to choose a unique solution for 11.53.

For $\gamma > 1$ (which means a high (absolute) value of the elasticity of marginal utility, that is, a low intertemporal substitution elasticity, the constant positive price solution is (locally) stable for all solution trajectories starting from a positive initial price of pet rocks. This is shown in Figure 11.3a.

Figure 11.3a here

For $\gamma < 1$, the constant positive price solution is unstable. The zero price equilibrium is locally stable for all solution trajectories starting from an initial price below the constant positive price solution. This is shown in Figure 11.3b.

Figure 11.3b here

.From 11.40 it follows that no equilibrium can exist that has the price of pet rocks rising without bound, since this would cause the value of the wealth of the old to rise without bound. Consumption by the old would rise without bound and this would violate the economy-wide real resource constraint. Non-stationary equilibria in which the price of pet rocks falls steadily and tends towards zero are possible, however. In the limit the economy would have a zero price of pet rocks.

Note that for pet rocks to have a positive price, it is necessary that there be an infinite number of generations. Per rocks, which are not intrinsically valuable as consumption goods, capital goods or intermediate inputs, have a positive value in period t only if it is expected to have a positive value in period $t+1$. It can only have a positive value in period $t+1$ if it is expected to have a positive value in period $t+2$ etc. Necessary for a positive value of pet rocks is therefore that following any generation, there is always another generation. This means that following any generation there will always be infinitely many generations. With generations living for a finite period of time, this implies that the economy has to go on forever. The key requirement, however, is that the number of generations be an open set, not that the economic system have an infinite lifetime.

For reasons I don't understand, some authors refer to the pet rocks of this section as *money*. The reciprocal of the price of pet rocks, $\frac{1}{P_t^M} \equiv P_t$ then is interpreted as the general price level. It should be clear that the pet rocks are nothing but a pure store of value, not a form of fiat money but a fiat asset. They do not serve as a means of payment, medium of exchange or transactions medium, for which there is no role in the model. They need not be the numéraire., unit of account or invoice unit, which is arbitrary in this model. If this is a theory of money, I am the emperor of China.

11.1.4 Intergenerational lending and borrowing with a richer OLG structure.

The 2-period model has no private intergenerational trade unless there are institutions that can enforce contracts between non-overlapping generations. Government is one such institution. Many different kinds of enterprises and corporate entities, that have legal personality and are capable of entering into contracts, are not tied, as regards the duration of these contracts by the life-span of the current owners or managers of the firm. It is, however, important to note that even without institutions that can bridge the contractual chasm between non-overlapping generations, the incomplete market participation problem gets less acute when more generations overlap.

Consider e.g. the case where $\pounds = 3$.

The budget constraints of the representative household of generation t are

$$c_t^1 + \tau_t^1 + \ell_t^1 + \sigma_t^1 \leq e_t^1 \tag{11.54}$$

$$c_t^2 + \tau_t^2 + \ell_t^2 + \sigma_t^2 \leq e_t^2 + (1 + r_{t+1})\ell_t^1 + (1 + \pi_{t+1})\sigma_t^1 \tag{11.55}$$

$$c_t^3 + \tau_t^3 + \ell_t^3 \leq e_t^3 + (1 + r_{t+2})\ell_t^2 + (1 + \pi_{t+2})\sigma_t^2 \tag{11.56}$$

$$\ell_t^3 \geq 0 \tag{11.57}$$

The loan market in any given period, t, now has the young, the middle-aged and the old participating in it. Clearly, the young can enter into contracts with the middle-aged, since both will be around the next period to service the loan.

Ignoring corner solutions, loan market equilibrium is characterised by

$$\ell_t^1 N_t + \ell_{t-1}^2 N_{t-1} + \ell_{t-2}^3 N_{t-2} = 0 \tag{11.58}$$

With the old not participating, this simplifies to

$$\ell_t^1 N_t + \ell_{t-1}^2 N_{t-1} = 0 \tag{11.59}$$

The intergenerational loan market can therefore be active for $\pounds \geq 3$.

11.1.5 Saving, capital accumulation and intergenerational redistribution

The material in this section is largely cribbed from Diamond [1965].

We retain the 2-period OLG structure and the CES utility function. Each household now receives an endowment of 1 unit of labour when young and nothing when old. The young again own a productive storage technology. They either sell their labour services or use their own labour services and/or those of other young households to work with capital goods that they rent

from the old, paying the old a rental rate ρ per unit of capital. The wage rate is w. Their will be no direct intergenerational lending between the generations, for the familiar reasons. If there were any heterogeneity among the young, there could be lending and borrowing among them. In a competitive economy, the rental rate of capital services is the marginal product of capital and the wage rate equals the marginal product of labour. Saving (disposable income minus consumption) by the young can either be allocated to the accumulation of real capital (to be leased next period to the next generation) or to government debt. The real interest rate equals the marginal product of capital. Saving by generation t when young is denoted s_t^1, and saving by generation t when old is denoted s_t^2. Real capital carried forward by a young household in period t is denoted $k_{t,t+1}^d$ and government debt carried forward is denoted $d_{t,t+1}^d$. The budget constraints of a household of generation t and the saving definitions are given in equations 11.60 to 11.62

$$w_t - \tau_t^1 - c_t^1 \equiv s_t^1 = k_{t.t+1}^d + d_{t.t+1}^d \tag{11.60}$$

$$c_t^2 + \tau_t^2 \equiv (1 + \rho_{t+1})k_{t,t+1}^d + (1 + r_{t+1})d_{t,t+1}^d \tag{11.61}$$

$$s_t^2 \equiv \rho_{t+1}k_{t,t+1}^d + r_{t+1}d_{t,t+1}^d - \tau_t^2 - c_t^2 \tag{11.62}$$

Note that

$$s_t^1 + s_t^2 = 0 \tag{11.63}$$

$$\rho_{t+1} = r_{t+1} \tag{11.64}$$

We can collapse the two single-period budget constraints into a single life-time budget constraint:

$$w_t - \tau_t^1 - \frac{\tau_t^2}{1 + r_{t+1}} = c_t^1 + \frac{c_t^2}{1 + r_{t+1}} \tag{11.65}$$

This is a trivial implication of the life-cycle model without intergenerational gifts and bequests. Each household dissaves during old age what it saves during its youth. We also define the following:

Aggregate saving by the young in period t, $S_t^1 \equiv N_t s_t^1$
Aggregate saving by the old in period t, $S_{t-1}^2 \equiv N_{t-1} s_{t-1}^2$
Aggregate private saving in period t, $S_t^P \equiv S_t^1 + S_{t-1}^2$
Government saving in period t, S_t^G
Aggregate national saving in period t, $S_t \equiv S_t^P + S_t^G$
Note that aggregate private saving can be written as follows:

$$\begin{aligned} S_t^P &= N_t s_t^1 - N_{t-1} s_{t-1}^1 \\ &= N_t (s_t^1 - s_{t-1}^1) + (N_t - N_{t-1})\, s_{t-1}^1 \\ &= N_t (s_t^1 - s_{t-1}^1) + n_t N_{t-1} s_{t-1}^1 \end{aligned} \tag{11.66}$$

Aggregate private saving increases either because the young are saving more than the old are dissaving or because there are more young than old.

In a steady state without growth (of population and productivity), total private saving will therefore be zero. Private financial wealth, A, can vary, however, even if steady state saving does not. The private wealth held at the end of period t (the beginning of period $t+1$), A_{t+1}, equals the saving of the young during period t

$$A_{t+1} = s_t^1 N_t \tag{11.67}$$

Private financial wealth in a closed economy is the sum of the capital stock, K, and the stock of public debt held by the private sector, D

$$A_{t+1} = K_{t+1} + D_{t+1} \tag{11.68}$$

Note that 11.67 and 11.68 are equivalent to the familiar goods market equilibrium condition (G is the volume of public consumption spending):

$$K_{t+1} - K_t + c_t^1 N_t + c_{t-1}^2 N_{t-1} + G_t = w_t N_t + \rho_t K_t$$

Competitive households take the wage rate, interest rate and capital rental rate as given. They also take taxes as given. Households maximize the utility function 11.6 subject to the budget constraints 11.60 and 11.61 and imposing 11.64.

The interior first-order condition is again:

$$\left(c_t^1\right)^{-\gamma} = \frac{1 + r_{t+1}}{1 + \delta} \left(c_t^2\right)^{-\gamma} \tag{11.69}$$

This implies the consumption functions

$$c_t^1 = \eta_t^1 \left(w_t - \tau_t^1 - \frac{\tau_t^2}{1 + r_{t+1}} \right) \tag{11.70}$$

$$c_t^2 = \eta_t^2 \left(w_t - \tau_t^1 - \frac{\tau_t^2}{1 + r_{t+1}} \right) \tag{11.71}$$

The marginal propensities to consume out of wealth are given by

$$\begin{aligned} \eta_t^1 &= \frac{(1+r_{t+1})^{\frac{\gamma-1}{\gamma}}}{(1+r_{t+1})^{\frac{\gamma-1}{\gamma}} + (1+\delta)^{-\frac{1}{\gamma}}} \\ &= \left(\frac{1+\delta}{2+\delta}\right) \quad \text{if } \gamma = 1 \end{aligned} \tag{11.72}$$

$$\begin{aligned} \eta_t^2 &= \left(\frac{(1+\delta)^{-\frac{1}{\gamma}}(1+r_{t+1})}{(1+r_{t+1})^{\frac{\gamma-1}{\gamma}} + (1+\delta)^{-\frac{1}{\gamma}}} \right) \\ &= \left(\frac{1+r_{t+1}}{2+\delta}\right) \quad \text{if } \gamma = 1 \end{aligned} \tag{11.73}$$

With the isoelastic utility function, consumption in each period is a normal good, that is,

$$0 < \eta_t^1 = 1 - \frac{\eta_t^2}{1 + r_{t+1}} < 1 \tag{11.74}$$

It is easily checked that

$$\frac{\partial \eta_t^1}{\partial(1 + r_{r+1})} = \left(\frac{\gamma - 1}{\gamma}\right) \frac{[(1 + r_{r+1})(1 + \delta)]^{-\frac{1}{\gamma}}}{\left[(1 + r_{t+1})^{\frac{\gamma-1}{\gamma}} + (1 + \delta)^{-\frac{1}{\gamma}}\right]^2}$$

Therefore the marginal (and average) propensity to consume out of wealth increases (decreases) when the real interest rate increases if and only if $\gamma > 1$ ($\gamma < 1$). Thus a higher real interest rate boosts the household propensity to save when young if and only if the intertemporal substitution elasticity, γ^{-1}, exceeds 1.

Note that the household's optimal consumption programme makes sense only if the government's fiscal policy is not so extravagant as to push the private sector into negative consumption, that is, we assume

$$w_t - \tau_t^1 - \frac{\tau_t^2}{1 + r_{t+1}} > 0. \tag{11.75}$$

We assume that the production technology is a neoclassical production function in capital and labour that has constant returns to scale, is strictly concave, three times continuously differentiable and satisfies the Inada conditions. The level of labour-augmenting technical efficiency is E, K is the capital stock and $\tilde{k} \equiv \frac{K}{EN}$ the capital stock per unit of labour measured in efficiency units.

$$\begin{array}{l} Y = F(K, EN) = ENF(\frac{K}{EN}, 1) = ENf(\frac{K}{EN}) \\ f(0) = 0; f' > 0; f'' < 0; \lim_{\tilde{k} \to 0} f'(\tilde{k}) = \infty; \lim_{\tilde{k} \to \infty} f'(\tilde{k}) = 0 \end{array}$$

Competitive capital rental markets and labour markets imply

$$r_{t+1} = f'(\tilde{k}_{t+1}) \tag{11.76}$$

$$w_t = E_t \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] \tag{11.77}$$

Labour efficiency is growing at the exogenous proportional rate ε

$$E_{t+1} = (1 + \varepsilon_{t+1}) E_t \tag{11.78}$$

The government spends G on current public consumption (there is no public sector capital formation), raises taxes from the young and the old, pays interest on its outstanding deficit and finances its financial deficit by issuing debt

$$D_{t+1} \equiv (1 + r_t) D_t + G_t - \tau_t^1 N_t - \tau_{t-1}^2 N_{t-1} \tag{11.79}$$

We define the following stocks and flows per unit of efficiency labour:

$$\begin{array}{r} \tilde{d}_{t+1} \equiv \frac{D_{t+1}}{E_{t+1} N_{t+1}} \\ \tilde{a}_{t+1} \equiv \frac{A_{t+1}}{E_{t+1} N_{t+1}} \\ . \quad \tilde{g}_t \equiv \frac{G_t}{E_t N_t} \\ \tilde{\tau}_t^1 \equiv \frac{\tau_t^1}{E_t} \\ \tilde{\tau}_t^2 \equiv \frac{\tau_t^2}{E_t} \end{array}$$

The government budget identity can now be rewritten as follows:

$$(1+\varepsilon_{t+1})(1+n_{t+1})\tilde{d}_{t+1} \equiv (1+r_t)\tilde{d}_t + \tilde{g}_t - \tilde{\tau}_t^1 - \frac{\tilde{\tau}_{t-1}^2}{(1+\varepsilon_t)(1+n_t)} \quad (11.80)$$

The financial market equilibrium condition can be rewritten as

$$(1+\varepsilon_{t+1})(1+n_{t+1})(\tilde{d}_{t+1}+\tilde{k}_{t+1}) = \frac{s_t^1}{E_t} = \frac{w_t - \tau_t^1 - c_t^1}{E_t}$$

or

$$\begin{aligned}&(1+\varepsilon_{t+1})(1+n_{t+1})(\tilde{d}_{t+1}+\tilde{k}_{t+1})\\ &= f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) - \tilde{\tau}_t^1 - \left(\frac{\left(1+f'(\tilde{k}_{t+1})\right)^{\frac{\gamma-1}{\gamma}}}{\left(1+f'(\tilde{k}_{t+1})\right)^{\frac{\gamma-1}{\gamma}} + (1+\delta)^{-\frac{1}{\gamma}}}\right)\left(f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) - \tilde{\tau}_t^1 - \frac{\tilde{\tau}_t^2}{\left(1+f'(\tilde{k}_{t+1})\right)}\right)\end{aligned} \quad (11.81)$$

Chapter 12

Private Investment

12.1 Fixed Investment

Investment is any activity involving costs and returns that are not perfectly synchronised in a single period. Many examples involve current costs and future returns, but even expected net returns (returns net of costs) could vary over time in a variety of ways. Sometimes the costs are fully sunk, that is, irrecoverable. At other times the initial outlays can be recouped to a greater or lesser extent, say through the sale of second-hand capital equipment. Fixed investment is the use of real resources to add to the fixed capital stock, plant, structures, equipment etc. Fixed investment can involve 'weightless' or intangible goods, e.g. the purchase of business software. As long as a capital good has an economic life beyond the current accounting period, adding to the stock of that capital good constitutes fixed investment. Households engage in fixed investment when they add to the stock of consumer durables. This is not counted as investment in the national income accounts. Households also add to the stock of human capital through education, training and other forms of learning. This too is not captured by the national accounts definition of investment.

12.2 Inventory Investment

Inventory investment involves changes in the stock of raw materials and intermediate inputs held either at the beginning of the production chain, or in the production chain (goods in process), as well as changes in the stock of finished goods. Inventories can be held by producers, wholesalers, retailers or households. Household inventory accumulation (changes in what we have in the fridge and on the kitchen shelves) is not counted as inventory

accumulation in the national accounts. Instead it is counted as consumer expenditure.

12.3 Neoclassical models of investment

12.3.1 Internal adjustment costs

For useful references see [30], [26] and [18].

A firm that does not own but rents its capital

Consider a competitive profit-maximising firm that produces a single output, Y, at a competitive price, P, using labour, L and capital, K. The money wage is W and the nominal rental rate of capital is R^K. The instantaneous nominal interest rate is i, and the capital depreciation rate δ. Output is produced using a well-behaved, constant returns to scale neoclassical production function, $Y = AF(K,L) = ALF(\frac{K}{L},1) = ALf(k)$, where $k \equiv \frac{K}{L}; f(0) = 0; f' > 0; f'' < 0$. The level of total factor productivity is $A > 0$.While capital is rented, the firm using the capital services incurs installation or adjustment costs whenever the capital stock is varied. These internal adjustment costs are quadratic in net investment.

The firm therefore maximises $\tilde{V}$, the present discounted value of current and future cash flow,

$$\tilde{V}(K(t),t) = \int_t^{\infty} e^{-\int_t^s i(u)du}[P(s)Y(s) - W(s)L(s) - R^K(s)K(s) - \frac{\gamma}{2}\frac{\dot{K}(s)^2}{K(s)}]ds \tag{12.1}$$

by suitable choices of employment and net capital investment at each point of time.

The Hamiltonian for this dynamic optimisation problem can be written as follows:

$$\begin{aligned}\tilde{H}(s) &= e^{-\int_t^s i(u)du}\begin{bmatrix} P(s)A(s)L(s)f\left(\frac{K(s)}{L(s)}\right) - W(s)L(s) \\ -R^K(s)K(s) - \frac{\gamma}{2}\frac{(I(s)-\delta K(s))^2}{K(s)} \end{bmatrix} \\ &\quad +\tilde{\lambda}(s)[I(s) - \delta K(s)]\end{aligned} \tag{12.2}$$

where $\tilde{\lambda}$ is the co-state variable of the capital stock, or its (present discounted value) shadow price. Note that $\tilde{\lambda}(t)$ is the shadow price of installed capital at time t, when the firm does not own the installed capital.

Among the first-order conditions for an optimum are

$$\frac{\partial \tilde{H}(s)}{\partial L(s)} = 0 \tag{12.3}$$

$$\frac{\partial \tilde{H}(s)}{\partial I(s)} = 0 \tag{12.4}$$

$$-\frac{\partial \tilde{H}(s)}{\partial K(s)} = \frac{d\tilde{\lambda}(s)}{ds} \tag{12.5}$$

Other boundary conditions include an initial condition for the capital stock

$$K(t) = \bar{K}(t) \tag{12.6}$$

and the condition that the present discounted value of the discounted capital stock in the infinite distant future be zero

$$\lim_{s\to\infty} \tilde{\lambda}(s)K(s) = 0 \tag{12.7}$$

Let $\tilde{q}(s)$ be the current value shadow price of capital for a firm that does not own its capital, that is,

$$\tilde{q}(s) = \tilde{\lambda}(s)e^{\int_t^s i(u)du} \tag{12.8}$$

These imply

$$A(s)\left(f\left(\frac{K(s)}{L(s)}\right) - \frac{K(s)}{L(s)}f'\left(\frac{K(s)}{L(s)}\right)\right) = \frac{W(s)}{P(s)} \tag{12.9}$$

$$\dot{K}(s) = I(s) - \delta K(s) = \frac{1}{\gamma}\tilde{q}(s)K(s) \tag{12.10}$$

$$\frac{d\tilde{q}(s)}{ds} = i(s)\tilde{q}(s) - \frac{1}{2\gamma}\tilde{q}(s)^2 + R^K(s) - P(s)A(s)f'\left(\frac{K(s)}{L(s)}\right) \tag{12.11}$$

A firm that owns its capital

Now consider the case of a firm that does not rent its capital but purchases it instead, at a price (net of installation costs), P^K. The present value of its current and future cash flow is

$$V(K(t),t) = \int_t^{\infty} e^{-\int_t^s i(u)du}[P(s)Y(s) - W(s)L(s) - P^K(s)I(s) - \frac{\gamma}{2}\frac{\dot{K}(s)^2}{K(s)}]ds \tag{12.12}$$

Its Hamiltonian is

$$\begin{aligned} H(s) \quad &= \quad e^{-\int_t^s i(u)du} \left[\begin{array}{c} P(s)L(s)A(s)f\left(\frac{K(s)}{L(s)}\right) - W(s)L(s) \\ -P^K(s)I(s) - \frac{\gamma}{2}\frac{[I(s)-\delta K(s)]^2}{K(s)} \end{array} \right] \\ &\quad +\lambda(s)[I(s) - \delta K(s)] \end{aligned} \tag{12.13}$$

The boundary conditions for this case are

$$K(t) = \bar{K}(t) \tag{12.14}$$

$$\lim_{s\to\infty} \lambda(s)K(s) = 0 \tag{12.15}$$

The current value shadow price of capital for this problem, q, is defined by $q(s) = \lambda(s)e^{\int_t^s i(u)du}$.

The first-order conditions for an optimum are 12.9 and

$$\dot{K}(s) = I(s) - \delta K(s) = \frac{1}{\gamma}[q(s) - P^K(s)]K(s) \tag{12.16}$$

$$\frac{dq(s)}{ds} = i(s)q(s) + \delta P^K(s) - P(s)A(s)f'\left(\frac{K(s)}{L(s)}\right) - \frac{1}{2\gamma}[q(s) - P^K(s)]^2 \tag{12.17}$$

In a world without transaction costs, a firm would be indifferent between renting its capital or owning it. The same real equilibrium should be supported under the two arrangements. Clearly this requires the following:

$$\tilde{q} = q - P^K \tag{12.18}$$

$$\frac{R^K}{P^K} - \delta + \frac{\dot{P}^K}{P^K} = i \tag{12.19}$$

Both of these are easily interpreted intuitively. The shadow price of capital when you don't own the capital, $\tilde{q}$, should obviously equal the shadow price of capital when you own it, q, minus its resale value, P^K. What would the competitive rental rate of capital, R^K be equal to? The instantaneous rate of return on 1$ invested in capital, when a unit of capital costs P^K would be $\frac{R^K}{P^K} - \delta + \frac{\dot{P}^K}{P^K}$. This would have to equal the opportunity cost of investing in capital, that is, the rate of interest.

The q of the model where the firm owns its capital is sometimes called Tobin's marginal q. Like any co-state variable, it satisfies the following condition

$$q(t) = \frac{\partial V(t)}{\partial K(t)} \tag{12.20}$$

Shadow prices are not directly observable. In a stock market economy, we observe Tobin's average q, denoted $\bar{q}$, the present discounted value of future cash flow, per unit of capital. That is,

$$\bar{q}(t) = \frac{V(t)}{K(t)} \tag{12.21}$$

When both the production function and the installation cost function are linear homogeneous, it so happens (check this!) that Tobin's marginal q is the same as Tobin's average q.

When there are no internal adjustment costs, the capital stock adjusts instantaneously to equate the marginal product of capital, at all times, to its competitive rental rate. That is, when $\gamma = 0$,

$$q = P^K \tag{12.22}$$

or

$$\tilde{q} = 0 \tag{12.23}$$

and

$$APf'\left(\frac{K}{L}\right) = R^K \tag{12.24}$$

Equation 12.19 continues to hold, so it remains true that the value of a unit of capital is the present discounted value of future capital rental income:

$$P^K(t) = q(t) = \int_t^{\infty} e^{-\int_t^s [i(u)+\delta]du} R^K(s)ds \tag{12.25}$$

A Tobin's q exercise Consider the system of equations 12.9, 12.16 and 12.17. From 12.9, we can express the capital-labour ratio, $\frac{K}{L}$ as a function of the real wage rate and the level of total factor productivity: $A[f(k) - kf'(k)] = \frac{W}{P}$. In what follows, I assume that $\frac{W}{P}$, A, P^K, P and i are all constant. Without loss of generality, I define units such that $\frac{W}{P} = A = P^K = P = 1$. Denote the constant marginal product of capital by $\alpha > 0$, that is, $f'(k) = \alpha$.

We can summarize the dynamics of the capital stock and its shadow price as follows:

$$\dot{K} = \frac{1}{\gamma}(q-1)K \tag{12.26}$$

$$\dot{q} = iq - \frac{1}{2\gamma}(q-1)^2 + \delta - \alpha \tag{12.27}$$

For there to exist a steady state with $q = 1$, it must be the case, that the three exogenous parameters, i, α and δ, satisfy the following relationship:

$$\alpha - \delta = i \tag{12.28}$$

Assume 12.28 is satisfied. This permits us to rewrite 12.27 as

$$\frac{d(q-1)}{dt} = i(q-1) - \frac{1}{2\gamma}(q-1)^2 \tag{12.29}$$

This univariate system has two steady states, $q = 1$ and $q = 1 + 2\gamma i$. The $q = 1$ steady state is unstable, the $q = 1 + 2\gamma i$ steady state is locally stable. This is shown in Figure 12.1. I assume that $i > 0$.

Figure 12.1 here

Figure 12.2 shows the dynamics of the capital stock and the shadow price of capital.

Figure 12.2 here

The isoclines for the capital stock (the combinations of q and K that support a constant level of the capital stock) are given by $K = 0$ and $q = 1$. It should be clear that a solution converging to $K = 0$ from a positive initial value of the capital stock cannot be optimal. We rule out any solution involving explosive negative values of q, since it would violate 12.15.

Assume we start at time 0 with a predetermined capital stock $K_0 > 0$. Any initial value of q below $q = 1$, will cause the capital stock to contract towards zero and to value of q to become negative. $q = 1$ is clearly a solution. The capital stock and q would be constant. What about an initial value of q above 1? Over time, q would converge to $1 + 2\gamma i$, either from above or from below. The capital stock would grow without bound. This is possible because the marginal product of capital is exogenously given (the firm can always hire enough labour to keep the capital-labour ratio constant at a given real wage. However, in steady state, the growth rate of the capital stock would be $\frac{\dot{K}}{K} = 2i$. The growth rate of the present value shadow price of capital would be $\frac{\dot{\lambda}}{\lambda} = -i$, since $q(s) = \lambda(s)e^{\int_t^s i(u)du}$ and, in steady state, q would be constant. Therefore, condition 12.15 would be violated. The only optimal solution therefore is $q = 1$.

What would happen if, instead, employment, L were exogenously given, say, $L = \bar{L} = 1$, say. In that case, the marginal product of capital would in, general, be decreasing in the capital stock. For simplicity, assume the marginal product of capital is linear in the capital stock: $Af'(K) = \alpha_0 - \alpha_1 K,\ \alpha_0,\, \alpha_1 > 0$.[1]

We can now write the two dynamical equations as follows

$$\dot{K} = \frac{1}{\gamma}(q - 1)K \tag{12.30}$$

$$\dot{q} = iq - \frac{1}{2\gamma}(q - 1)^2 + \delta - \alpha_0 + \alpha_1 K \tag{12.31}$$

[1] The production function implied by this is $Y = \alpha_0 K - \frac{1}{2}\alpha_1 \frac{K^2}{L}$

The steady state of this model is given by

$$q = 1$$

$$K = \frac{\alpha_0 - \delta - i}{\alpha_1} \tag{12.32}$$

Equation 12.32 makes sense only if $\alpha_0 - \delta - i > 0$.

The $\dot{K} = 0$ locus is the horizontal line $q = 1$ (for $K \geq 0$) and the vertical line $K = 0$ (for $q > 0$). The $\dot{q} = 0$ locus is given by

$$q = 1 + \gamma i \pm \sqrt{(i\gamma)^2 + 2\gamma[i - (\alpha_0 - \delta - \alpha_1 K)]}$$

Figure 12.3 shows the equilibrium configuration in $q - K$ space. The unique steady state, Ω, is a saddlepoint. Note that K is a predetermined state variable and q a non-predetermined state variable.

Figure 12.3 here

Figure 12.4 shows the dynamic response of the system to an unanticipated, immediate and permanent increase in the interest rate, i. We consider the linearized system in the neighbourhood of the steady state:

$$\begin{bmatrix} \dot{q} \\ \dot{K} \end{bmatrix} \approx \begin{bmatrix} \bar{\imath} - \frac{\bar{q}-1}{\gamma} & \alpha_1 \\ \frac{\bar{K}}{\gamma} & \frac{\bar{q}-1}{\gamma} \end{bmatrix} \begin{bmatrix} q - \bar{q} \\ K - \bar{K} \end{bmatrix} + \begin{bmatrix} \bar{q} \\ 0 \end{bmatrix} [i - \bar{\imath}] \tag{12.33}$$

or

$$\begin{bmatrix} \dot{q} \\ \dot{K} \end{bmatrix} \approx \begin{bmatrix} \bar{\imath} & \alpha_1 \\ \frac{\alpha_0 - \delta - \bar{\imath}}{\alpha_1 \gamma} & 0 \end{bmatrix} \begin{bmatrix} q - \bar{q} \\ K - \bar{K} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [i - \bar{\imath}] \tag{12.34}$$

We assume $\bar{\imath} > 0$, so the $\dot{q} = 0$ locus is downward-sloping. Since $\bar{K} > 0$, the equilibrium is a saddlepoint.

Figure 12.4 here

An unanticipated immediate permanent increase in the rate of interest causes an immediate drop in q from Ω_0 to $\Omega_{01.}$ From there on the system travels smoothly along the convergent saddlepath through Ω_1 to the new steady state.

12.3.2 External adjustment costs

Without internal adjustment costs, and without further modifications of the model, the capital stock adjusts instantaneously (through infinite rates of investment or disinvestment) to equate the marginal product of capital to its competitive rental rates. Even without adjustment costs, one can retain finite investment rates, at any rate at the level of the economy as a whole, by postulating that the capital goods producing industry is subject to increasing marginal costs.

$$P^K(t) = \int_t^\infty e^{-\int_t^s [i(u)+\delta]du} R^K(s)ds \tag{12.35}$$

$$R^K = PAf'\left(\frac{K}{L}\right) \tag{12.36}$$

$$\begin{aligned} P^K &= \sigma_0 + \sigma_1(I - \delta K) \\ \sigma_{0,}\sigma_1 &> 0 \end{aligned} \tag{12.37}$$

$$\dot{K} = I - \delta K \tag{12.38}$$

The supply curve of the capital goods producing industries is 12.37. Assume again that employment is given exogenously at $L = 1$, that $A = P = 1$ and that $f'(K) = \alpha_0 - \alpha_1 K,\ \alpha_0,\ \alpha_1 > 0$.

We get the following dynamic system for the capital stock and the price of capital

$$\dot{K} = \frac{1}{\sigma_1}(P^K - \sigma_0) \tag{12.39}$$

$$\dot{P}^K = (i + \delta)P^K - \alpha_0 + \alpha_1 K \tag{12.40}$$

Note that this had exactly the same structure as 12.34.

$$\begin{bmatrix} \dot{P}^K \\ \dot{K} \end{bmatrix} \approx \begin{bmatrix} \bar{\imath} + \delta & \alpha_1 \\ \frac{1}{\sigma_1} & 0 \end{bmatrix} \begin{bmatrix} P^K \\ K \end{bmatrix} + \begin{bmatrix} -\alpha_0 \\ -\frac{\sigma_0}{\sigma_1} \end{bmatrix} \tag{12.41}$$

Consider the special case where people have static expectations. In this case,

$$P^K = \frac{\alpha_0 - \alpha_1 K}{i + \delta} \tag{12.42}$$

Figure 12.5 here

Figure 12.5 shows, in the LHS diagram, the demand curve for the capital stock and the historically given stock of capital, $K(0)$. This determines the equilibrium price of capital, $P^K(0)$. The supply curve of new capital goods in the RHS diagram then determines the rate of capital formation. Assume, say, that this is positive. The vertical supply curve of capital (the capital stock) shifts to the right. This lowers the future price of capital. The system reaches equilibrium when the price has fallen to the point where net capital formation is zero (see [29]).

Exercise 12.3.1 *Consider the following economy with external costs of adjusting the capital stock. P^K is the price of capital, i the short nominal interest rate, δ the depreciation rate, R^K the rental rate of capital, K the capital stock, I gross investment and τ the tax rate on profits.*

$$P^K(t) = \int_t^\infty e^{-\int_t^s [i(u)+\delta]du} R^K(s)ds \tag{12.43}$$

$$\begin{aligned} R^K &= (1-\tau)(\alpha_0 - \alpha_1 K) && (12.44)\\ \alpha_0,\, \alpha_1 &> 0; 0 < \tau < 1 && (12.45) \end{aligned}$$

$$\begin{aligned} P^K &= \sigma_0 + \sigma_1(I - \delta K) && (12.46)\\ \sigma_0, \sigma_1 &> 0 \end{aligned}$$

$$\dot{K} = I - \delta K \tag{12.47}$$

(I) Analyse the effect on the capital stock and on the price of capital of the unexpected announcement, at $t = t_0$, of an immediate, permanent increase in the tax rate on profits.

(II)Analyse the effect on the capital stock and on the price of capital of the unexpected announcement, at $t = t_0$, of a future permanent increase in the tax rate on profits, starting at $t_1 > t_0$.

The two investment models considered thus far had convex production technologies and convex costs of adjusting the capital stock (costs that increase more than linearly with the rate of investment). The result is, not surprisingly, that investment is 'smoothed out'. In the next section, I will consider optimal investment when the cost of adjusting the capital stock is 'lumpy'. There are fixed, and sunk (that is, irrecoverable) costs of expanding or contracting the capital stock. Again unsurprisingly, such a technology leads to investment being bunched, rather than smoothed out. When uncertainty is thrown into the pot, interesting interpretations of optimal investment rules, involving option pricing, can be made.

12.3.3 Investment under uncertainty with sunk costs of expanding or contracting the capital stock

The material for this subsection is straight from Abel, Dixit, Eberly and Pindyck [1995] [1] (see also [12]). Consider a firm that has a two-period horizon. In period 1 the firm buys capital, $K_1 \geq 0$, at a price $b > 0$ per unit of capital. The gross operating return in period 1 is known and given by $r(K_1)$, which is positive, strictly increasing, strictly concave in K_1 and twice continuously differentiable.. Capital does not depreciate.

In period 2, the firm can operate the same level of capital as in period 1 (that is, $K_2 = K_1$), neither buying nor selling capital in period 2. It can also choose to operate a smaller capital stock in period 2 than in period 1. If it chooses to do so, it has to sell its 'excess capital' in period 2 at a price b_L. If $b_L < (1 + r)b$, where r is the rate of interest, reversing the period 1 investment is costly. We assume that this condition holds. Alternatively, the firm can choose to operate a larger capital stock in period 2 than in period 1. In that case it has to purchase the additional capital in period 2 at a price b_H. If $b_H > (1 + r)b$, expanding the capital stock in period 2 is costly. We assume that this condition holds. Note that therefore, $b_H > b_L$.

The gross return to operating the capital stock in period 2 is $R(K_2, \varepsilon)$, where ε is random. An increase in ε represents 'good news' to the firm. Think of it as a positive productivity shock or a positive shock to its price. The gross marginal product of capital in period 2, $R_K(K_2, \varepsilon) \geq 0$ is twice continuously differentiable, strictly decreasing in K, and strictly increasing in ε, that is, $R_{KK} < 0$ and $R_{K\varepsilon} > 0$.

The firm is risk-neutral and maximises the expected present discounted

value of current and expected future profits. Let $V(K_1)$ denote the expected present discounted value of profits in periods 1 and 2, when the firm has a capital stock K_1 in period 1. Then the period 1 decision problem of the firm is

$$\max_{K_1}\{V(K_1) - bK_1\} \tag{12.48}$$

We will now analyse how uncertainty about the future returns to investment affects the period 1 investment decision of the firm.

We start by defining two critical values, ε_L and ε_H, of the exogenous profitability shock, ε.

$$R_K(K_1, \varepsilon_L) = b_L \tag{12.49}$$

$$R_K(K_1, \varepsilon_H) = b_H \tag{12.50}$$

Note that, because we assumed $b_H > b_L$ and $R_{K\varepsilon} > 0$, it will be the case that $\varepsilon_H > \varepsilon_L$.

In period 2, following the realisation of the random shock ε, the capital stock will be adjusted to its optimal level, $K_2(\varepsilon)$.

If $\dot{\varepsilon} > \varepsilon_H$, the firm will purchase additional capital until the marginal product of capital in period 2 equals its period 2 purchase price, that is, until

$$R_K(K_2(\varepsilon), \varepsilon) = b_H \tag{12.51}$$

If $\dot{\varepsilon} < \varepsilon_L$, the firm will sell capital until the marginal product of capital in period 2 equals its period 2 sale price, that is, until

$$R_K(K_2(\varepsilon), \varepsilon) = b_L \tag{12.52}$$

If $\varepsilon_L \leq \varepsilon \leq \varepsilon_H$, the firm will neither purchase not sell capital, so

$$K_2(\varepsilon) = K_1 \tag{12.53}$$

This means that there is a 'zone of inaction': for some intermediate range of values of the profitability shock, the firm will neither contract nor expand. The zone of inaction is illustrated in Figure 12.6.

Figure 12.6 here

Let $F(\varepsilon)$ be the cumulative distribution function of the profitability shock, that is,

$$F(e) \equiv= \text{prob}(\varepsilon \leq e)$$

and $dF(\varepsilon)$ its probability density function.

We can express $V(K_1)$, the expected present value of period 1 and period 2 profits if the firm starts with a capital stock K_1, as follows:

$$\begin{aligned} V(K_1) &= r(K_1) \\ &+ \frac{1}{1+r}\left\{\begin{array}{l} \int_{-\infty}^{\varepsilon_L}\left[R\left(K_2(\varepsilon),\varepsilon\right)+b_L\left(K_1-K_2(\varepsilon)\right)\right]dF(\varepsilon) \\ +\int_{\varepsilon_L}^{\varepsilon_H}R\left(K_1,\varepsilon\right)dF(\varepsilon) \\ +\int_{\varepsilon_H}^{\infty}\left[R\left(K_2(\varepsilon),\varepsilon\right)+b_H\left(K_1-K_2(\varepsilon)\right)\right]dF(\varepsilon) \end{array}\right\} \end{aligned} \tag{12.54}$$

The first-order conditions for the optimisation of 12.48 is therefore

$$V'(K_1) = b$$

or

$$V'(K_1) = r'(K_1) + \frac{1}{1+r}\left\{b_L F(\varepsilon_L) + \int_{\varepsilon_L}^{\varepsilon_H} R_K\left(K_1,\varepsilon\right)dF(\varepsilon) + b_H[1-F(\varepsilon_H)]\right\} = b \tag{12.55}$$

Tobin's marginal q is given by

$$q^M(K_1) = \frac{V'(K_1)}{b} \tag{12.56}$$

Tobin's average q is given by

$$q^A(K_1) = \frac{V(K_1)}{bK_1} \tag{12.57}$$

We denote the marginal valuation of capital in period 1, $V'(K_1)$ by $q(K_1)$. It follows from 12.49 to 12.53, that the first-order condition 12.55 can be rewritten as follows:

$$q(K_1) = V'(K_1) = r'(K_1) + \frac{1}{1+r}\int_{-\infty}^{\infty} R_K\left(K_2(\varepsilon), \varepsilon\right) dF(\varepsilon) = b \quad (12.58)$$

The marginal valuation of capital in period 1 is the marginal revenue product of capital in period 1, plus the expected present discounted value of the marginal revenue product of capital in period 2, where the marginal revenue product of capital is evaluated at the optimal level of the capital stock in period 2. The marginal revenue product of capital in period 2 is the resale value of capital in period 2 for low realisations of the profitability shock, that is, for $\varepsilon < \varepsilon_L$; it is $R_K(K_1, \varepsilon)$ for intermediate realisations of the profitability shock when the firm neither buys nor sells capital and operates the period 1 capital stock, that is for $\varepsilon_L \leq \varepsilon \leq \varepsilon_H$, and it is the purchase price of additional capital in period 2 when the firm expands the capital stock in period 2, for high realisations of the profitability shock, that is, for $\varepsilon > \varepsilon_L$.

Note that

$$q'(K_1) = V''(K_1) = r''(K_1) + \frac{1}{1+r}\int_{\varepsilon_L}^{\varepsilon_H} R_{KK}\left(K_1, \varepsilon\right) dF(\varepsilon) < 0 \quad (12.59)$$

Because the marginal revenue product of capital is diminishing in the capital stock, the following comparative static results follow immediately:

$$\frac{\partial K_1}{\partial b} = \frac{1}{q'} < 0 \quad (12.60)$$

$$\frac{\partial K_1}{\partial b_L} = -\left(\frac{1}{1+r}\right)\frac{F(\varepsilon_L)}{q'} > 0 \quad (12.61)$$

$$\frac{\partial K_1}{\partial b_H} = -\left(\frac{1}{1+r}\right)\frac{[1 - F(\varepsilon_H)]}{q'} > 0 \quad (12.62)$$

Obviously, an increase in the period 1 purchase price of capital reduces the demand for period 1 capital. However, both future *increased reversibility* (a higher resale price of capital, b_L) and future *reduced expandability* (a higher future purchase price of capital, b_H) increase the marginal valuation of period 1 capital and therefore raise the optimal period 1 capital stock.

An option interpretation of the optimal investment rule

We can rewrite $V(K_1)$ as follows:

$$V(K_1) = N(K_1) + \frac{1}{1+r}P(K_1) - \frac{1}{1+r}C(K_1) \tag{12.63}$$

where

$$N(K_1) = r(K_1) + \frac{1}{1+r}\left\{\int_{-\infty}^{\infty} R\left(K_1, \varepsilon\right) dF(\varepsilon)\right\} \tag{12.64}$$

$$P(K_1) = \int_{-\infty}^{\varepsilon_L} \left[R\left(K_2(\varepsilon), \varepsilon\right) - R\left(K_1, \varepsilon\right) + b_L\left(K_1 - K_2(\varepsilon)\right)\right] dF(\varepsilon) \tag{12.65a}$$

$$C(K_1) = -\left\{ \int_{\varepsilon_H}^{\infty} \left[R\left(K_2(\varepsilon), \varepsilon\right) - R\left(K_1, \varepsilon\right) + b_H\left(K_1 - K_2(\varepsilon)\right)\right] dF(\varepsilon) \right\} \tag{12.66}$$

Here $N(K_1)$ is the value of the capital installed in period 1, if it were impossible to change the capital stock in period 2.

$P(K_1)$ is the value of the *put option* to sell (any or all of the capital installed in period 1 during period 2 at a *strike price* b_L. A put option is the right (not the obligation) to sell something at (sometimes at or before) a certain date at a prearranged price, the strike price. Note that this put option will only be exercised if $\varepsilon < \varepsilon_L$.

$C(K_1)$ is the value of the *call option* to buy, in period 2, any amount of capital in excess of the amount bought in period 1 (K_1) at a *strike price* b_H. A call option is the right (not the obligation) to sell something at (sometimes at or before) a certain date at a prearranged price, the strike price. Note that this call option will only be exercised if $\varepsilon > \varepsilon_H$.

Thus, when you buy durable capital in period 1, you really are engaged in three transactions. You buy capital that you will use in period 1 and, come hell or high water (that is for all realisations of ε) in period 2. You also acquire the right to sell any or all of K_1 during period 2 at the resale price b_L. This is the put option you acquire when you purchase a unit of durable capital in period 1. Finally, by purchasing K_1 units of capital during period 1, you give up the right to buy these K_1 units instead during period 2, at a price b_H. That is, you extinguish a call option to buy the durable capital good later, when you buy the durable capital good today.

The first-order condition for an optimum, 12.55 can now be rewritten as follows:

$$V'(K_1) = N'(K_1) + \frac{1}{1+r}P'(K_1) - \frac{1}{1+r}C'(K_1) = b \tag{12.67}$$

where

$$N'(K_1) = r'(K_1) + \frac{1}{1+r}\left\{\int_{-\infty}^{\infty} R_K\left(K_1,\varepsilon\right) dF(\varepsilon)\right\} \tag{12.68}$$

$$\begin{aligned} P'(K_1) &= \textstyle\int_{-\infty}^{\varepsilon_L}\left[b_L - R_K\left(K_1,\varepsilon\right)\right] dF(\varepsilon) && \text{(12.69a)} \\ &= E\left\{\max\left[b_L - R_K\left(K_1,\varepsilon\right),0\right]\right\} && \text{(12.69b)} \end{aligned}$$

$$\begin{aligned} C'(K_1) &= \textstyle\int_{\varepsilon_H}^{\infty}\left[R_K\left(K_1,\varepsilon\right) - b_H\right] dF(\varepsilon) && \text{(12.70)} \\ &= E\left\{\max[R_K\left(K_1,\varepsilon\right) - b_H, 0\right\} && \text{(12.71)} \end{aligned}$$

$N'(K_1)$ is the expected present discounted value of the marginal revenue products of capital evaluated at the initial, period 1, capital stock, K_1.

$P'(K_1)$ is the value of a marginal put option on the capital in period 1, that is, the value of an option to sell an additional unit of capital in period 2, at a strike price b_L, with the option evaluated at the optimal period 1 capital stock; $C'(K_1)$ is the value of a marginal call option on the capital in period 1, that is the value of an option to buy an additional unit of capital in period 2 at a strike price b_H, with the option evaluated at the optimal period 1. capital stock.

$N'(K_1)$ can be interpreted as the "naive" present discounted value of current and future marginal revenue products of capital, which assumes that the period 2 capital stock will be the same as the period 1 capital stock. We will use the notation

$$\begin{aligned} n(K_1) &= N'(K_1) && \text{(12.72)} \\ c(K_1) &= C'(K_1) \\ p(K_1) &= P'(K_1) \end{aligned}$$

The first-order condition for investment in period 1 can therefore be rewritten as:

$$q(K_1) = n(K_1) + \frac{1}{1+r}p(K_1) - \frac{1}{1+r}c(K_1) = b \qquad (12.73)$$

We can easily derive the impact of changes in the purchase price of period 2 capital on c and of changes in the sale price of period 2 capital on p.

$$\frac{\partial c}{\partial b_H} = -[1 - F(\varepsilon_H)] \leq 0$$

$$\frac{\partial p}{\partial b_L} = F(\varepsilon_L)] \geq 0$$

Investment and uncertainty about future returns

A 'better' return distribution We compare two return distributions, $F(\varepsilon)$ and $H(\varepsilon)$. First, F is assumed to 'first-order' dominate H. What this means is the following (see [1] Hirshleifer and Riley [1992], pp. 105-117.). We assume all distribution functions are differentiable at least one more time than we need.

Definition 3 ***First-order stochastic dominance***
If, for all e, $F(e) \leq H(e)$ and the inequality is strict for some interval, then the distribution F exhibits first-order stochastic dominance over H.

Thus, in Figure 12.7, F and J first-order stochastically dominate H , but F does not first-order stochastically dominate J or vice-versa.

Figure 12.7 here

This definition leads immediately to the following theorem:

Theorem 4 ***Ranking Theorem 1***
For all increasing, piecewise differentiable functions $x(e)$, if F exhibits first-order stochastic dominance over H, then $E_F\{x(e)\} > E_H\{x(e)\}$.

Let M be the heavy line function in Figure 12.6, that is,

$$\begin{aligned} M(K_1, \varepsilon) &= b_L \qquad \text{if } \varepsilon < \varepsilon_L \\ &= R_K(K_1, \varepsilon) \qquad \text{if } \varepsilon_L \leq \varepsilon \leq \varepsilon_H \\ &= b_H \qquad \text{if } \varepsilon > \varepsilon_H \end{aligned}$$

It follows that

$$q(K_1) = r'(K_1) + \frac{1}{1+r}\left\{\int_{-\infty}^{\infty} M\left(K_1, \varepsilon\right) dF(\varepsilon)\right\}$$

This shows that q is the expected value of a non-decreasing function of ε. Therefore, a first-order shift to the right in the distribution of ε cannot lower this expected value. The incentive to invest in period 1 is therefore not lowered on balance.

Moreover, the function M is strictly increasing in ε in the intermediate range $\varepsilon_L \leq \varepsilon \leq \varepsilon_H$, and takes on constant values to the left of ε_L and to the right of ε_H. Unless the shift of the distribution is entirely confined to the ranges $(-\infty, \varepsilon_L]$ and $[\varepsilon_H, \infty)$, the incentive to invest is actually increased.

Thus, if the (very) bad news becomes worse, or if the (very) good news becomes better, this does not affect the incentive to invest. q is affected by the cumulative probability to the left of $\varepsilon_{L,}$ $F(\varepsilon_L)$ and by the cumulative probability to the right of $\varepsilon_{H,}$ $(1 - F(\varepsilon_H))$, but not by any details of the probability densities in the two tails. When things were bad and simply get worse, you sell more capital at a fixed price. When things were good and simply get better, you buy more capital at a fixed price. The incentive to invest is affected by the details of the probability density function in the intermediate range only.

A more uncertain return distribution Now consider the case where F second-order stochastically dominates H.

Definition 5 ***Second-order stochastic dominance***
If, for all e,

$$\int_a^e F(\varepsilon)d\varepsilon \leq \int_a^e H(\varepsilon)d\varepsilon$$

with the inequality holding strictly over some part of the range, then the distribution F exhibits second-order stochastic dominance over H.

Geometrically, F is second-order dominant over H if, over every interval $[a, e]$ the area under $F(e)$ is never greater (and sometimes smaller) than the corresponding area under $H(e)$. In terms of Figure 12.7, this would require the horizontally shaded area to be larger than the vertically shaded area.

Theorem 6 ***Ranking Theorem 2***

For all increasing concave twice piecewise-differentiable functions $x(e)$, the concavity being strict somewhere, if F exhibits second-order stochastic dominance over H, then $E_F\{x(e)\} > E_H\{x(e)\}$.

If the two distributions have the same mean, F being second-order stochastically dominant over H, is a formalisation of the notion that H is more risky than F. In fact, H will have a higher variance than F.

An immediate implication of Theorem 2 is the following:

Theorem 7 ***Ranking Theorem 3***

For all increasing concave twice piecewise-differentiable functions $x(e)$, the concavity being strict somewhere, if F and H have the same mean and F exhibits second-order stochastic dominance over H, then $E_F\{x(e)\} > E_H\{x(e)\}$.

A second-order shift in the distribution function of ε, e.g. a mean preserving spread, has an ambiguous effect on the incentive to invest. We consider the effect of increased uncertainty about future returns on $n(K_1)$, on $p(K_1)$ and on $c(K_1)$ in turn.

Consider 12.73. The effect of increased uncertainty on the "naive" present discounted value of current and future marginal revenue products of capital, which assumes that the period 2 capital stock will be the same as the period 1 capital stock, can be eyeballed from the following equation.

$$n(K_1) = r'(K_1) + \frac{1}{1+r}\left\{\int_{-\infty}^{\infty} R_K\left(K_1, \varepsilon\right) dF(\varepsilon)\right\} \tag{12.74}$$

A larger variance increases (decreases) $n(K_1)$ if the marginal product of capital in period 2 is a convex (concave) function of ε. This is a straightforward implication of *Jensen's inequality.*

Convex Functions

A real-valued function g defined on an interval (a, b) of the real line is convex on (a, b) if, for any two points x_1 and x_2 in (a, b) and any number μ such that $0 < \mu < 1$

$$g[\mu x_1 + (1-\mu)x_2] \leq \mu g(x_1) + (1-\mu)g(x_2) \tag{12.75}$$

Geometrically, this means that the line segment joining any two points on the curve $x_2 = g(x_1)$ does not fall below the curve anywhere between those two points. The function g is strictly convex on (a, b) if strict equality

holds in the relation 12.75 for every pair of distinct points x_1 and x_2 in (a, b). Geometrically, this means that the function g is convex and that the curve $x_2 = g(x_1)$ cannot contain any linear segments. A function g is (strictly) concave on (a, b) if the negative of g is (strictly) convex on (a, b). In Figure 12.8, the first function is strictly convex and the second function is convex but not strictly convex, because it contains linear segments.

More generally,

Let g be a convex function, then, for any $x_1, ..., x_n$

$$\begin{aligned} g\left(\sum_{i=1}^{n} \mu_i x_i\right) &\leq \left(\sum_{i=1}^{n} \mu_i g(x_i)\right) \\ \mu_i &\geq 0; \sum_{i=1}^{n} \mu_i = 1 \end{aligned} \tag{12.76}$$

If g is a strictly convex function, the inequality in 12.76 is strict.

Figure 12.8 here

Jensen's inequality relates to expectations of convex functions, and follows immediately from 12.76.

Theorem 8 ***Jensen's inequality***

Let g be a convex function on the interval (a, b), and let X be a random variable such that $\Pr\{X \in (a, b)\} = 1$ *and the expectations* $E\{X\}$ *and* $E\{g(X)\}$ *exist. Then*

$$E\{g(X)\} \geq g[E\{X\}] \tag{12.77}$$

Furthermore, if g is strictly convex and $\Pr\{X = E\{X\}\} \neq 1$, *then there is a strict inequality in relation 12.77.*

It is not too difficult to think of profitability shocks that might cause the marginal revenue product of capital to increase more than linearly with ε, that is, the case in which $R_K(K_1, \varepsilon)$ is an (increasing and) strictly convex function of ε. In this case, n, which is the expectation of $R_K(K_1, \varepsilon)$,would, by Jensen's inequality, increase with an increase in the variance of ε.

What about the effect of an increase in uncertainty on the value of the two marginal options? A key intuition about options is that their value increases with the level of uncertainty. Both the put option that is acquired when a

unit of capital is purchased in period 1 and the call option that is extinguished by the purchase of capital in period 1 would therefore be expected to increase with an increase in uncertainty. The net effect on the incentive to invest in period 1 would be positive if the value of the put option were to increase more than the value of the call option, that is, if the increase in uncertainty is concentrated in the lower tail rather than in the upper tail.

$$p(K_1) = \int_{-\infty}^{\varepsilon_L} [b_L - R_K(K_1,\varepsilon)]\, dF(\varepsilon) \tag{12.78a}$$

$$c(K_1) = \int_{\varepsilon_H}^{\infty} [R_K(K_1,\varepsilon) - b_H]\, dF(\varepsilon) \tag{12.79}$$

Figure 12.9 illustrates the principles at work. Let $x = R_K(K_1,\varepsilon) = \rho(\varepsilon)$ be the marginal revenue product of capital in period 2, (with the fixed period 1 capital stock), and let $\Phi(x)$ be the cumulative distribution function of x. The density function of x, $\varphi(x) = d\Phi(x)$ is induced by the density function of ε, $f(\varepsilon) = dF(\varepsilon)$ as follows:

Let ε be a continuous random variable having density $f(\varepsilon)$, defined on the interval $I = (-\infty, \infty)$. Let ρ be a differentiable strictly increasing[2] or strictly decreasing function on the interval I and let $\rho(I)$ denote the range of ρ and ρ^{-1} the inverse function to ρ.Then $x = R_K(K_1,\varepsilon) = \rho(\varepsilon)$ has density φ given by

$$\begin{aligned} \varphi(x) &= f(\varepsilon)\left|\frac{dx}{d\varepsilon}\right| \\ x &\in \rho(I) \\ \varepsilon &= \rho^{-1}(x) \end{aligned}$$

Note that [3]

$$\begin{aligned} p(K_1) &\equiv \int_{-\infty}^{\varepsilon_L} [b_L - R_K(K_1,\varepsilon)]\, dF(\varepsilon) = \int_0^{b_L} [b_L - x]\, d\Phi(x) \\ &= [b_L - x]\, \Phi(x)|_0^{b_L} + \int_0^{b_L} \Phi(x)dx = \int_0^{b_L} \Phi(x)dx \end{aligned}$$

[2] In fact, we assume that the marginal revenue product of capital is continuous and strictly increasing in ε. We also assumed $R_K(K_1,\varepsilon) \geq 0$, (and that it is strictly decreasing and continuous in K, two properties that do not matter here, since K_1 is fixed).

[3] Using integration by parts:

$$\int_a^b x_1 dx_2 = x_1 x_2|_a^b - \int_a^b x_2 dx_1$$

and

$$
\begin{aligned}
c(K_1) &\equiv \int_{\varepsilon_H}^{\infty} [R_K(K_1, \varepsilon) - b_H]\, dF(\varepsilon) = \int_{b_H}^{\infty} [x - b_H]\, d\Phi(x) \\
&= [x - b_H]\,[\Phi(x) - 1]|_{b_H}^{\infty} - \int_{b_H}^{\infty} [\Phi(x) - 1]\, dx = \int_{b_H}^{\infty} [1 - \Phi(x)]\, dx
\end{aligned}
$$

Figure 12.9 here

The value of the marginal put option is the shaded area below b_L. The value of the marginal call option is the shaded area above b_H. An increase in risk twists the pdf of x in a clockwise direction. If the 'pivot point' is in the range between b_L and b_H, both shaded areas will increase.

The conventional wisdom that increased uncertainty increases the 'option value of waiting', and thus reduces investment, focuses only on the increase in the value of the call option that is forfeited when investment occurs in period 1. That is, it has emphasized irreversibility (a very low sale price of capital, b_L in period 2). If, by contrast, capital investment were easily reversed (b_L is high), but very costly to expand in the future (b_H is high), investing today would be more attractive if uncertainty about future returns increased.

Chapter 13

Money

Chapter 14

A Benchmark for Monetary Policy

The concept of a neutral monetary policy, and of a neutral short-term nominal interest rate is an intriguing one.

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